

# F-PURITY VERSUS LOG CANONICITY FOR POLYNOMIALS

DANIEL J. HERNÁNDEZ

**ABSTRACT.** In this article, we consider the conjectured relationship between  $F$ -purity and log canonicity for polynomials over  $\mathbb{C}$ . We associate to a collection  $\mathcal{M}$  of  $n$  monomials a rational polytope  $\mathbf{P}$  contained in  $[0, 1]^n$ . Using  $\mathbf{P}$  and the Newton polyhedron associated to  $\mathcal{M}$ , we define a non-degeneracy condition under which log canonicity and dense  $F$ -pure type are equivalent for all  $\mathbb{C}^*$ -linear combinations of the monomials in  $\mathcal{M}$ . We also show that log canonicity corresponds to  $F$ -purity for very general polynomials. Our methods rely on showing that the  $F$ -pure and log canonical threshold agree for infinitely primes, and we accomplish this by comparing these thresholds with the thresholds associated to their monomial ideals.

## INTRODUCTION

Let  $f \in \mathbb{L}[x_1, \dots, x_m]$  be a polynomial over a field of characteristic  $p > 0$  with  $f(\mathbf{0}) = 0$ . Let  $\mathbf{m}^{[p^e]} := (x_1^{p^e}, \dots, x_m^{p^e})$  denote the  $e^{\text{th}}$  Frobenius power of  $\mathbf{m} := (x_1, \dots, x_m)$ . The limit

$$(0.0.1) \quad \mathbf{fpt}_{\mathbf{m}}(f) = \lim_{e \rightarrow \infty} \frac{\max \{ r : f^r \notin \mathbf{m}^{[p^e]} \}}{p^e}$$

exists, and is called the  $F$ -pure threshold of  $f$  at  $\mathbf{m}$  [HW02, TW04, MTW05]. This invariant measures of the singularities of  $f$  near  $\mathbf{0}$ , and is closely related to the theory of  $F$ -purity and tight closure [HR76, HH90, HY03]. If  $\mathcal{M}$  is a collection of monomials, one may also define the  $F$ -pure threshold of the ideal generated by  $\mathcal{M}$ , denoted  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M})$ , and if  $f$  is a  $\mathbb{L}^*$ -linear combination of the elements of  $\mathcal{M}$ , one has the inequality

$$(0.0.2) \quad \mathbf{fpt}_{\mathbf{m}}(f) \leq \min \{ 1, \mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) \}.$$

This inequality may be strict:  $\mathbf{fpt}_{\mathbf{m}}(u \cdot x^p + v \cdot y^p) = \frac{1}{p}$ , while  $\mathbf{fpt}_{\mathbf{m}}(x^p, y^p) = \frac{2}{p}$ .

Suppose that  $\mathcal{M}$  consists of  $n$  monomials in  $m$  variables. In Definition 3.3, we associate to  $\mathcal{M}$  a rational polytope  $\mathbf{P}$  contained in  $[0, 1]^n$  called the *splitting polytope* of  $\mathcal{M}$ . The polytope  $\mathbf{P}$  is closely related to the familiar *Newton polyhedron*  $\mathbf{N} \subseteq \mathbb{R}^m$  associated to  $\mathcal{M}$  and the geometry of each determines the value of  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M})$ . We call  $\boldsymbol{\eta} \in \mathbf{P}$  *maximal* if the sum of the entries of  $\boldsymbol{\eta}$  is maximal among all coordinate sums of elements of  $\mathbf{P}$  (see Definition 3.4). In this article, we identify conditions on  $\mathbf{N}$ ,  $\mathbf{P}$  and  $p$  under which equality holds in (0.0.2). We summarize some of these results below. In what follows, we assume that  $f \in \mathbb{L}[x_1, \dots, x_m]$  and is a  $K^*$ -linear combination of the monomials in  $\mathcal{M}$ .

- (1) If  $\mathbf{P}$  contains a unique maximal point  $\boldsymbol{\eta}$ , then equality holds in (0.0.2) whenever  $(p-1) \cdot \boldsymbol{\eta} \in \mathbf{N}$ . This is a direct corollary of the Theorem 4.2, which also produces a lower bound for  $\mathbf{fpt}_{\mathbf{m}}(f)$  when equality does not hold (0.0.2).

---

The author was partially supported by the National Science Foundation RTG grant number 0502170 at the University of Michigan.

- (2) Suppose that  $\min \{1, \mathbf{fpt}_m(\mathcal{M})\} = \mathbf{fpt}_m(\mathcal{M})$ . If  $(p-1) \cdot \mathbf{fpt}_m(\mathcal{M}) \in \mathbb{N}$ , Proposition 3.21 states there exists a closed set  $Z \subseteq \mathbb{A}_K^n$  such that equality holds in (0.0.2) whenever the coefficients of  $f$  are not in  $Z$ . The condition  $(p-1) \cdot \mathbf{fpt}_m(\mathcal{M}) \in \mathbb{N}$  is necessary, as we saw in the example following (0.0.2).
- (3) Suppose instead that  $\min \{1, \mathbf{fpt}_m(\mathcal{M})\} = 1$ . In the two dimensional case, the failure of equality to hold in (0.0.2) allows one to produce an explicit upper bound on  $p$ . This statement is a corollary of Lemma 3.23, which gives the same conclusion in higher dimensions modulo some additional hypotheses and generalizes [Fed83, Lemma 2.3].

We now switch gears and consider an invariant of singularities for polynomials defined over  $\mathbb{C}$ . Let  $f$  be a polynomial over  $\mathbb{C}$  with  $f(\mathbf{0}) = 0$ . If  $\lambda > 0$ , the function  $\frac{1}{|f|^\lambda}$  has a pole at  $\mathbf{0}$ , and understanding how “bad” this pole is provides a measure of the singularities of  $f$  at  $\mathbf{0}$ . In particular, one may ask whether this function is  $L^2$ , which leads us to the definition of the *log canonical threshold* of  $f$ , denoted  $\mathbf{lct}_0(f)$ :

$$\mathbf{lct}_0(f) = \sup \left\{ \lambda : \frac{1}{|f|^{2\lambda}} \text{ is locally integrable at } \mathbf{0} \right\}.$$

Log canonical thresholds can also be defined using information obtained via (log) resolution of singularities, and this invariant plays an important role in higher dimensional birational geometry [BL04, Laz04]. Remarkably,  $F$ -pure thresholds can be thought of as the positive characteristic analog of log canonical thresholds. [Smi00, HW02, HY03, Tak04]. We now briefly sketch the relationship between these two invariants. If  $f$  has rational coefficients, one may reduce them modulo  $p$  to obtain polynomials  $f_p$  over the finite fields  $\mathbb{F}_p$  for  $p \gg 0$ . Otherwise, one still obtains a *family of positive characteristic models*  $f_p$  over finite fields of characteristic  $p$  via the process of *reduction to positive characteristic*. Using the results of [HY03], it is observed in [MTW05] that

$$(0.0.3) \quad \mathbf{fpt}_m(f_p) \leq \mathbf{lct}_0(f) \text{ for } p \gg 0, \text{ and that } \lim_{p \rightarrow \infty} \mathbf{fpt}_m(f_p) = \mathbf{lct}_0(f).$$

**Example 0.1.** If  $f = x^2 + y^3$ , then  $\mathbf{lct}_0(f) = \frac{5}{6}$ , and

$$\mathbf{fpt}_m(f_p) = \begin{cases} 1/2 & p = 2 \\ 2/3 & p = 3 \\ 5/6 & p \equiv 1 \pmod{6} \\ \frac{5}{6} - \frac{1}{6p} & p \equiv 5 \pmod{6} \end{cases}.$$

We say that *log canonicity equals dense  $F$ -purity* for  $f$  whenever  $\mathbf{fpt}_m(f_p) = \mathbf{lct}_0(f)$  for infinitely many  $p$ ; see Remark 5.11 for a justification of this terminology. Note that log canonicity equals dense  $F$ -purity for  $x^3 + y^2$ , as  $p \equiv 1 \pmod{6}$  for infinitely many primes.

**Example 0.2.** Let  $f \in \mathbb{Q}[x, y, z]$  be a form of degree 3 with isolated singularity at  $\mathbf{0}$ , so that  $f$  defines an elliptic curve  $E \subseteq \mathbb{P}^2$ . Then,  $\mathbf{lct}_0(f) = 1$ , and  $\mathbf{fpt}_m(f_p) = 1$  if and only if  $E_p = \mathbb{V}(f_p)$  is not *supersingular*. That log canonicity equals dense  $F$ -purity in this example follows from work of Serre. [MTW05, Example 4.6].

It is conjectured that log canonicity equals dense  $F$ -purity for all polynomials, and verifying this correspondence is a long-standing open problem [Fed83, Smi97, EM06]. We now summarize the results in this article related to this correspondence.

- (4) Let  $\mathcal{M}$  again denote a collection of monomials, and suppose that its associated polytope  $\mathbf{P}$  contains a unique maximal point. Theorem 5.15 then states that log canonicity equals dense  $F$ -purity for every  $\mathbb{C}^*$ -linear combination of the monomials of  $\mathcal{M}$ .
- (5) In Theorem 5.16, we also see that log canonicity equals dense  $F$ -purity for all polynomials whose coefficients form an algebraically independent set over  $\mathbb{Q}$ .

We now briefly outline our methods for proving the statements in (4) and (5). As with  $F$ -pure thresholds, one may extend the definition of the log canonical threshold to any ideal vanishing at  $\mathbf{0}$ . If  $\mathcal{M}$  denotes a set of monomials and  $f$  is a  $\mathbb{C}^*$ -linear combination of the members of  $\mathcal{M}$ , then  $\mathbf{lct}_0(f) \leq \min\{1, \mathbf{lct}_0(\mathcal{M})\}$ . The value of  $\mathbf{fpt}_m(\mathcal{M})$  has a natural formula in terms of the geometry of the splitting polytope  $\mathbf{P}$ , while  $\mathbf{lct}_0(\mathcal{M})$  can be computed via the Newton polyhedron  $\mathbf{N}$ . On the other hand, the connection between these two polytopes allows one to conclude that  $\mathbf{fpt}_m(\mathcal{M}) = \mathbf{lct}_0(\mathcal{M})$ , a formula that also follows from the general statements in [HY03]. We continue to assume that  $f$  is a  $\mathbb{C}^*$ -linear combination of the members of  $\mathcal{M}$ . Using the results referenced in (1) and (2), we are able to show that  $\mathbf{fpt}_m(f_p) = \min\{1, \mathbf{fpt}_m(\mathcal{M})\}$  for infinitely many  $p$ , and applying the relations in (0.0.3) shows that for such  $p$ ,

$$\mathbf{fpt}_m(f) \leq \mathbf{lct}_0(f) \leq \min\{1, \mathbf{lct}_0(\mathcal{M})\} = \min\{1, \mathbf{fpt}_m(f)\} = \mathbf{fpt}_m(f),$$

forcing equality throughout.

**Acknowledgments.** Many of the results contained in this article appear in my Ph.D. thesis, completed at the University of Michigan. I would like to thank my advisor, Karen Smith, for all her guidance during this project. I am also grateful to Emily Witt, both for her support and for her assistance with many of the details in this article. I would also like to thank Mel Hochster for talking with me about  $F$ -purity, and Vic Reiner for answering some of my questions regarding convex geometry.

## 1. BASE $p$ EXPANSIONS

**Definition 1.1.** Let  $\alpha \in (0, 1]$ . A *non-terminating base  $p$  expansion* of  $\alpha$  is the unique expression of the form  $\alpha = \sum_{e \geq 1} \frac{a_e}{p^e}$  with the property that the integers  $a_e$  are in  $[0, p-1]$  and are all not eventually zero. The number  $a_e$  is called the  $e^{\text{th}}$  digit of  $\alpha$  in base  $p$ .

For example,  $1 = \sum_{e \geq 1} \frac{p-1}{p^e}$  is a non-terminating base  $p$  expansion of 1.

**Definition 1.2.** Let  $\alpha \in (0, 1]$ , and fix a prime  $p$ .

- (1)  $\alpha^{(e)}$  will always denote the  $e^{\text{th}}$  digit of  $\alpha$  in base  $p$ . By convention,  $\alpha^{(0)} = 0^{(e)} = 0$ .
- (2) We call  $\langle \alpha \rangle_d := \sum_{e=1}^d \frac{\alpha^{(e)}}{p^e}$  the  $e^{\text{th}}$  truncation of  $\alpha$  in base  $p$ .
- (3) For  $\alpha = (\alpha_1, \dots, \alpha_n) \in [0, 1]^n$ , we set  $\langle \alpha \rangle_e := (\langle \alpha_1 \rangle_e, \dots, \langle \alpha_n \rangle_e)$ .

**Terminology 1.3.** When referring to the  $e^{\text{th}}$  digit (respectively, truncation) of  $\alpha$ , we will always mean the  $e^{\text{th}}$  digit (respectively, truncation) in some base  $p$  which will always be obvious from the context (and will often be equal to the characteristic of an ambient field). This explains the absence of the base prime  $p$  in the notation  $\alpha^{(e)}$  and  $\langle \alpha \rangle_e$ .

**Lemma 1.4.** If  $(p^e - 1) \cdot \alpha \in \mathbb{N}$  for some  $e$ , then  $(p^e - 1) \cdot \alpha = p^e \cdot \langle \alpha \rangle_e$ .

*Proof.* This follows from the observation (whose verification is left to the reader) that if  $(p^e - 1) \cdot \alpha \in \mathbb{N}$ , then the digits of  $\alpha$  (in base  $p$ ) are periodic and repeat after  $e$  terms.  $\square$

**Lemma 1.5.** If  $\alpha \in (0, 1]$ , then  $\langle \alpha \rangle_e \in \frac{1}{p^e} \cdot \mathbb{N}$ , and  $\langle \alpha \rangle_e < \alpha \leq \langle \alpha \rangle_e + \frac{1}{p^e}$ . Furthermore, if  $(p-1) \cdot \alpha \in \mathbb{N}$ , then  $\alpha^{(e)} = (p-1) \cdot \alpha$  for every  $e \geq 1$ .

*Proof.* The first two assertions follow from the definitions. For the third, note that the non-terminating base  $p$  expansion for  $\alpha$  can be obtained by multiplying each term in the non-terminating base  $p$  expansion of  $1 = \sum_{e \geq 1} \frac{p-1}{p^e}$  by  $\alpha$ .  $\square$

**Definition 1.6.** Let  $(\alpha_1, \dots, \alpha_n) \in [0, 1]^n$ . We say *the digits of  $\alpha_1, \dots, \alpha_n$  add without carrying* (in base  $p$ ) if  $\alpha_1^{(e)} + \dots + \alpha_n^{(e)} \leq p-1$  for every  $e \geq 1$ . For  $(k_1, \dots, k_n) \in \mathbb{N}^n$ , we say that the digits of  $k_1, \dots, k_n$  *add without carrying* (in base  $p$ ) if the analogous condition holds for the digits in the unique base  $p$  expansions of the integers  $k_1, \dots, k_n$ .

**Remark 1.7.** We point out that the digits of  $\alpha_1, \dots, \alpha_n$  add without carrying if and only if the digits of the integers  $p^e \langle \alpha_1 \rangle_e, \dots, p^e \langle \alpha_n \rangle_e$  add without carrying for every  $e \geq 1$ .

The concept of adding without carrying is relevant in light of the following classical result.

**Lemma 1.8.** [Dic02, Luc78] Let  $(k_1, \dots, k_n) \in \mathbb{N}^n$ , and set  $N = \sum_i k_i$ . Then, the multinomial coefficient  $\binom{N}{\mathbf{k}} := \frac{N!}{k_1! \dots k_n!} \not\equiv 0 \pmod{p}$  if and only if the digits of  $k_1, \dots, k_n$  add without carrying (in base  $p$ ).

**Lemma 1.9.** Let  $(\alpha_1, \dots, \alpha_n) \in \mathbb{Q}^n \cap [0, 1]^n$ .

- (1) If  $\alpha_1 + \dots + \alpha_n \leq 1$ , there exist infinitely many primes  $p$  for which the digits of  $\alpha_1, \dots, \alpha_n$  add without carrying (in base  $p$ ).
- (2) Otherwise, there exist infinitely many primes  $p$  for which  $\alpha_1^{(1)} + \dots + \alpha_n^{(1)} \geq p$ .

*Proof.* By Dirichlet's theorem on primes in arithmetic progressions,  $(p-1) \cdot (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  for infinitely many primes  $p$ . For such primes, Lemma 1.5 shows that  $\alpha_i^{(e)} = (p-1) \cdot \alpha_i$  are the digits of  $\alpha_i$ . The lemma follows from this observation.  $\square$

## 2. $F$ -PURE THRESHOLDS

**Universal Hypothesis 2.1.** A field  $\mathbb{L}$  of prime characteristic  $p > 0$  is called  *$F$ -finite* if  $[\mathbb{L} : \mathbb{L}^p] < \infty$ . Throughout this article, all fields of prime characteristic will be assumed to be  $F$ -finite.

Let  $R = \mathbb{L}[x_1, \dots, x_m]$  denote the polynomial ring over a field of characteristic  $p > 0$ , and let  $f$  be a non-zero polynomial in  $R$  with  $f(\mathbf{0}) = 0$ , so that  $f \in \mathfrak{m} := (x_1, \dots, x_m)$ . For every  $I \subseteq R$ , let  $I^{[p^e]}$  denote the ideal generated by the set  $\{r^{p^e} : r \in I\}$ . We call  $I^{[p^e]}$  the  $e^{\text{th}}$  *Frobenius power* of  $I$ . Note that  $(I^{[p^e]})^{[p^\ell]} = I^{[p^{e+\ell}]}$ . It is well known that  $R$  is a finitely-generated, free  $R^{p^e}$ -module. Using this fact, it is easy to verify that  $g^{p^e} \in I^{[p^e]}$  if and only if  $g \in I$ . Consider the values

$$(2.0.4) \quad \nu_f(p^e) := \max \{a : f^a \notin \mathfrak{m}^{[p^e]}\}.$$

As  $f \in \mathfrak{m}$ , we have that  $f^{p^e} \in \mathfrak{m}^{[p^e]}$ , so that  $0 \leq \nu_f(p^e) \leq p^e - 1$ . As  $f^a \notin \mathfrak{m}^{[p^e]}$  implies that  $(f^a)^{p^d} = f^{p^d a} \notin \mathfrak{m}^{[p^{e+d}]}$ , we have that  $p^d \cdot \nu_f(p^e) \leq \nu_f(p^{e+d})$ . These observations show that the numbers  $\frac{\nu_f(p^e)}{p^e}$  define a non-decreasing sequence of non-negative rational numbers contained in the unit interval.

**Definition 2.2.** [TW04, MTW05] The limit  $\mathbf{fpt}_m(f) := \lim_{e \rightarrow \infty} \frac{\nu_f(p^e)}{p^e}$  exists, and is called the *F-pure threshold of  $f$  at  $\mathbf{m}$* .

**Remark 2.3.** As  $f \neq 0$ , there exists  $e \geq 1$  such that  $f \notin \mathbf{m}^{[p^e]}$ . Thus,  $\nu_f(p^e) \geq 1$ , and it follows that  $\mathbf{fpt}_m(f) \in (0, 1]$ .

**Remark 2.4.** The  $\nu_f(p^e)$  determine  $\mathbf{fpt}_m(f)$  by definition, but it turns out that the converse is true as well. Specifically, it is shown in [MTW05, Proposition 1.9] that

$$(2.0.5) \quad \nu_f(p^e) = \lceil p^e \cdot \mathbf{fpt}_m(f) \rceil - 1,$$

where  $\lceil \cdot \rceil$  denotes the least integer greater than function. For a generalization, see [Her10b].

**Lemma 2.5.** Let  $\lambda \in [0, 1]$  be a rational number such that  $(p^e - 1) \cdot \lambda \in \mathbb{N}$  for some  $e \geq 1$ . Then  $f^{(p^e-1)\cdot\lambda} \notin \mathbf{m}^{[p^e]}$  if and only if  $\mathbf{fpt}_m(f) \geq \lambda$ .

Lemma 2.5 is closely related to results from [Her10b] and [Sch08].

*Proof of Lemma 2.5.* First, suppose that  $\mathbf{fpt}_m(f) \geq \lambda$ . It follows from (2.0.5) that

$$\nu_f(p^e) \geq \lceil p^e \lambda \rceil - 1 = \lceil (p^e - 1) \cdot \lambda + \lambda \rceil - 1 = (p^e - 1) \cdot \lambda + \lceil \lambda \rceil - 1 = (p^e - 1) \cdot \lambda,$$

so that  $f^{(p^e-1)\cdot\lambda} \notin \mathbf{m}^{[p^e]}$ . We now address the other implication.

As  $\mathbf{m}^{[p^e]}$  is  $\mathbf{m}$ -primary, we have that  $\mathbf{m}^{[p^e]} \cdot R_m \cap R = \mathbf{m}^{[p^e]}$ . Thus, we may localize at  $\mathbf{m}$ , and assume that  $(R, \mathbf{m})$  is a local ring. Next, note that

$$(2.0.6) \quad (p^{e(d+1)} - 1) \cdot \lambda = p^e \cdot (p^{ed} - 1) \cdot \lambda + (p^e - 1) \cdot \lambda,$$

which shows that  $(p^{ed} - 1) \cdot \lambda \in \mathbb{N}$  for all  $d \geq 1$ . We now induce on  $d$  to show that

$$(2.0.7) \quad f^{(p^{ed}-1)\cdot\lambda} \notin \mathbf{m}^{[p^{ed}]} \text{ for every } d \geq 1,$$

the base case being our hypothesis. By means of contradiction, assume that (2.0.7) holds, but that  $f^{(p^{e(d+1)}-1)\cdot\lambda} \in \mathbf{m}^{[p^{e(d+1)}]}$ . Applying (2.0.6) shows that

$$(2.0.8) \quad f^{(p^e-1)\cdot\lambda} \in \left( \mathbf{m}^{[p^{e(d+1)}]} : f^{p^e(p^{ed}-1)\cdot\lambda} \right) = \left( \mathbf{m}^{[p^{ed}]} : f^{(p^{ed}-1)\cdot\lambda} \right)^{[p^e]} \subseteq \mathbf{m}^{[p^e]},$$

which is a direct contradiction of our initial hypothesis. Thus, (2.0.7) holds by induction, and implies that  $\frac{\nu_f(p^{ed})}{p^{ed}} \geq \frac{(p^{ed}-1)\cdot\lambda}{p^{ed}}$ . Finally, letting  $d \rightarrow \infty$  shows that  $\mathbf{fpt}_m(f) \geq \lambda$ .  $\square$

We can generalize the above setup as follows: Given any ideal  $\mathfrak{a} \subseteq \mathbf{m}$ , let

$$(2.0.9) \quad \nu_{\mathfrak{a}}(p^e) := \max \left\{ r : \mathfrak{a}^r \not\subseteq \mathbf{m}^{[p^e]} \right\}.$$

As before,  $\frac{\nu_{\mathfrak{a}}(p^e)}{p^e}$  defines a non-decreasing sequence of non-negative rational numbers. If  $\mathfrak{a}$  is generated by  $N$  elements, one can also check that  $\nu_{\mathfrak{a}}(p^e) \leq N(p^e - 1) + 1$ , so that  $\lim_{e \rightarrow \infty} \frac{\nu_{\mathfrak{a}}(p^e)}{p^e}$  exists, and is bounded above by  $N$ . We will call  $\lim_{e \rightarrow \infty} \frac{\nu_{\mathfrak{a}}(p^e)}{p^e}$  the *F-pure threshold of  $\mathfrak{a}$  at  $\mathbf{m}$* , and denote it by  $\mathbf{fpt}_m(\mathfrak{a})$ . As before, it is easy to see that  $\mathbf{fpt}_m(\mathfrak{a}) > 0$ .

**Notation 2.6.** We often write  $\mathbf{fpt}_m(f)$  and  $\mathbf{fpt}_m(\mathfrak{a})$  rather than  $\mathbf{fpt}_m(R, f)$  and  $\mathbf{fpt}_m(R, \mathfrak{a})$ . Furthermore, if  $\mathcal{N}$  is a subset of  $R$ , we will use  $\mathbf{fpt}_m(\mathcal{N})$  to denote the *F-pure threshold of the ideal generated by  $\mathcal{N}$* .

**Remark 2.7.** Note that if  $f \in \mathfrak{a}$ , then  $\mathbf{fpt}_m(f) \leq \mathbf{fpt}_m(\mathfrak{a})$ . Thus, if  $f$  is a  $K^*$ -linear combination of a set of monomials  $\mathcal{M}$ , it follows from this observation and Remark 2.3 that  $\mathbf{fpt}_m(f) \leq \min \{1, \mathbf{fpt}_m(\mathcal{M})\}$ . Also, although it is not obvious,  $\mathbf{fpt}_m(f)$  and  $\mathbf{fpt}_m(\mathfrak{a})$  are rational numbers [BMS08, Theorem 3.1].

### 3. SPLITTING POLYTOPES AND NEWTON POLYHEDRA

#### 3.1. On splitting polytopes.

**Notation 3.1.** For  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{R}^n$ ,  $|\mathbf{s}|$  will denote the coordinate sum  $s_1 + \dots + s_n$ . We stress that  $|\cdot|$  is *not* the usual Euclidean norm on  $\mathbb{R}^n$ . Furthermore, when dealing with elements of  $\mathbb{R}^n$ , we use  $\prec$  and  $\preceq$  to denote component-wise (strict) inequality. Finally,  $\mathbf{1}_m$  will denote the element  $(1, \dots, 1) \in \mathbb{R}^m$ .

Let  $\mathcal{M} := \{\mathbf{x}^{a_1}, \dots, \mathbf{x}^{a_n}\}$  be a collection of distinct monomials in  $x_1, \dots, x_m$ , where each exponent vector  $\mathbf{a}_i$  is a non-zero element of  $\mathbb{N}^m$ .

**Definition 3.2.** We call the  $m \times n$  matrix  $\mathbf{E} := (\mathbf{a}_1 \cdots \mathbf{a}_n)$  the *exponent matrix* of  $\mathcal{M}$ .

**Definition 3.3.** We call  $\mathbf{P} := \{\mathbf{s} \in \mathbb{R}_{\geq 0}^n : \mathbf{E}\mathbf{s} \preceq \mathbf{1}_m\}$  the *splitting polytope* of  $\mathcal{M}$ . As  $\mathbf{E}$  has non-negative integer entries,  $\mathbf{P}$  is contained in  $[0, 1]^n$ .

**Definition 3.4.** For  $\lambda \in \mathbb{R}_{\geq 0}$ , let  $\mathbf{P}_\lambda$  denote the hyperplane section  $\mathbf{P} \cap \{\mathbf{s} : |\mathbf{s}| = \lambda\}$ . If  $\alpha = \max \{|\mathbf{s}| : \mathbf{s} \in \mathbf{P}\}$ , we set  $\mathbf{P}_{\max} := \mathbf{P}_\alpha$ , and we call the elements of  $\mathbf{P}_{\max}$  the *maximal points* of  $\mathbf{P}$ .

**Example 3.5.** If  $\mathcal{M} = \{x_1^{d_1}, \dots, x_m^{d_m}\}$ , then  $\mathbf{P} = \left\{ \mathbf{s} \in \mathbb{R}^m : \mathbf{0} \preceq \mathbf{s} \preceq \left(\frac{1}{d_1}, \dots, \frac{1}{d_m}\right) \right\}$ , and  $\mathbf{P}_{\max} = \left\{ \left(\frac{1}{d_1}, \dots, \frac{1}{d_m}\right) \right\}$ .

**Example 3.6.** If  $\mathcal{M} = \{x^a, y^b, (xy)^c\}$ , then  $\mathbf{P} = \left\{ \mathbf{s} \in \mathbb{R}_{\geq 0}^3 : \begin{array}{l} as_1 + cs_3 \leq 1 \\ bs_2 + cs_3 \leq 1 \end{array} \right\}$  is the convex hull of  $\mathbf{v}_1 = (\frac{1}{a}, 0, 0)$ ,  $\mathbf{v}_2 = (0, \frac{1}{b}, 0)$ ,  $\mathbf{v}_3 = (\frac{1}{a}, \frac{1}{b}, 0)$ ,  $\mathbf{v}_4 = (0, 0, \frac{1}{c})$ , and  $\mathbf{0}$ .

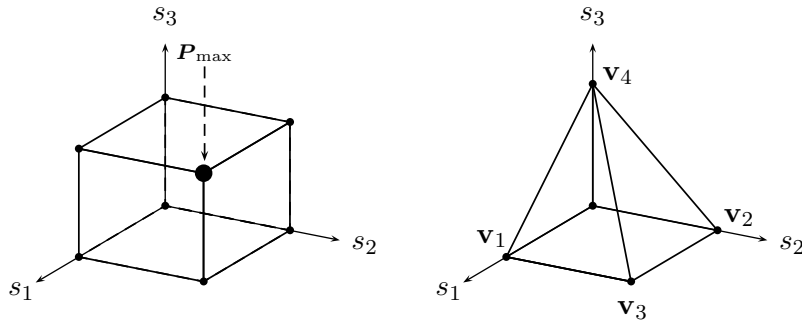


FIGURE 1. The splitting polytopes from Examples 3.5 and 3.6.

The following illustrates the connection between splitting polytopes and  $F$ -pure thresholds.

**Proposition 3.7.** Over any field of characteristic  $p > 0$ ,  $\mathbf{fpt}_m(\mathcal{M}) = \max \{|\mathbf{s}| : \mathbf{s} \in \mathbf{P}\}$ .



*Proof.* Note that the elements  $\mathbf{x}^{k_1 \mathbf{a}_1} \cdots \mathbf{x}^{k_n \mathbf{a}_n} = \mathbf{x}^{\mathbf{E} \mathbf{k}}$  with  $k_1 + \cdots + k_n = N$  generate  $(\mathcal{M})^N$ . Let  $\nu(p^e) := \nu_{(\mathcal{M})}(p^e)$  be as in (2.0.9). As  $(\mathcal{M})^{\nu(p^e)} \not\subseteq \mathfrak{m}^{[p^e]}$ , there exists  $\mathbf{k} \in \mathbb{N}^n$  with  $|\mathbf{k}| = \nu(p^e)$  such that  $\mathbf{x}^{\mathbf{E} \mathbf{k}} \notin \mathfrak{m}^{[p^e]}$ . By definition,  $\frac{1}{p^e} \cdot \mathbf{k} \in \mathbf{P}$ , and consequently  $\frac{\nu(p^e)}{p^e} = \frac{|\mathbf{k}|}{p^e} = \left| \frac{1}{p^e} \cdot \mathbf{k} \right|$ , which by definition is bounded above by  $\max \{ |\mathbf{s}| : \mathbf{s} \in \mathbf{P} \}$ . Taking  $e \rightarrow \infty$  then shows that  $\mathbf{fpt}_m(\mathcal{M}) \leq \max \{ |\mathbf{s}| : \mathbf{s} \in \mathbf{P} \}$ .

Next, choose  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ . As  $\langle \boldsymbol{\eta} \rangle_e \prec \boldsymbol{\eta}$ , we have that  $\mathbf{E} \langle \boldsymbol{\eta} \rangle_e \prec \mathbf{E} \boldsymbol{\eta} \preceq \mathbf{1}_m$ , and so  $\mathbf{x}^{p^e \mathbf{E} \langle \boldsymbol{\eta} \rangle_e}$  is contained in  $(\mathcal{M})^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$  but not  $\mathfrak{m}^{[p^e]}$ . It follows that  $\frac{\nu_a(p^e)}{p^e} \geq |\langle \boldsymbol{\eta} \rangle_e|$ , and again letting  $e \rightarrow \infty$  shows that  $\mathbf{fpt}_m(\mathcal{M}) \geq |\boldsymbol{\eta}| = \max \{ |\mathbf{s}| : \mathbf{s} \in \mathbf{P} \}$ .  $\square$

**3.2. On Newton polyhedra.** We next investigate the role of the Newton polyhedron associated to  $\mathcal{M}$ , and begin by reviewing some basic notions from convex geometry.

**Discussion 3.8.** Recall that  $\mathcal{P} \subseteq \mathbb{R}^m$  is called a *polyhedral set* if there exist finitely many linear forms  $L_1, \dots, L_d$  in  $\mathbb{R}[z_1, \dots, z_m]$  and elements  $\beta_1, \dots, \beta_d \in \mathbb{R}$  such that

$$(3.2.1) \quad \mathcal{P} = \{ \mathbf{v} \in \mathbb{R}^m : L_i(\mathbf{v}) \geq \beta_i \text{ for all } 1 \leq i \leq d \}.$$

$\mathcal{P}$  is called *rational* if  $L_i \in \mathbb{Q}[z_1, \dots, z_m]$  and  $\beta_i \in \mathbb{Q}$  for all  $1 \leq i \leq d$ .

Given any finite set  $\mathcal{C} \subseteq \mathbb{R}^m$ , we use  $\mathcal{C}^{\text{cone}}$  to denote  $\{ \sum_{\mathbf{v} \in \mathcal{C}} \lambda_{\mathbf{v}} \cdot \mathbf{v} : \lambda_{\mathbf{v}} \geq 0 \}$ , the cone generated by  $\mathcal{C}$ , and  $\mathcal{C}^{\text{convex}}$  to denote  $\{ \sum_{\mathbf{v} \in \mathcal{C}} \lambda_{\mathbf{v}} \cdot \mathbf{v} : \lambda_{\mathbf{v}} \geq 0 \text{ and } \sum_{\mathbf{v} \in \mathcal{C}} \lambda_{\mathbf{v}} = 1 \}$ , the convex hull of  $\mathcal{C}$ . Though it is not obvious from these definitions, both  $\mathcal{C}^{\text{cone}}$  and  $\mathcal{C}^{\text{convex}}$  are polyhedral sets; see [Web94, Theorem 4.1.1 and Theorem 3.2.5]. Additionally, these polyhedral sets are rational if and only if  $\mathcal{C} \subseteq \mathbb{Q}^m$ .

If  $L \in \mathbb{R}[z_1, \dots, z_m]$  is a non-zero linear form and  $\beta$  is a real number, we use  $H_L^\beta$  to denote  $\{ \mathbf{v} \in \mathbb{R}^m : L(\mathbf{v}) \geq \beta \}$ , the (upper) *halfspace* determined by  $L$  and  $\beta$ . If  $\mathcal{P}$  is a polyhedral set, then  $H_L^\beta$  is called a *supporting halfspace* of  $\mathcal{P}$  if  $\mathcal{P} \subseteq H_L^\beta$  and  $L^{-1}(\beta) \cap \mathcal{P} \neq \emptyset$ . In this case,  $L^{-1}(\beta) \cap \mathcal{P}$  is called an (exposed) *face* of  $\mathcal{P}$ . Given two subsets  $S$  and  $S'$  of  $\mathbb{R}^m$ , we will use  $S + S'$  to denote  $\{ \mathbf{v} + \mathbf{w} : \mathbf{v} \in S, \mathbf{w} \in S' \}$ , the Minkowski sum of  $S$  and  $S'$ . A set in  $\mathbb{R}^m$  is polyhedral if and only if it is of the form  $\mathcal{C}^{\text{convex}} + \Gamma^{\text{cone}}$ , where  $\mathcal{C}$  and  $\Gamma$  are two finite (and possibly empty) subsets of  $\mathbb{R}^m$  [Web94, Theorem 4.1.2].

In what follows, we abuse notation and use  $\mathcal{M}^{\text{convex}}$  to denote  $\{ \mathbf{a}_1, \dots, \mathbf{a}_n \}^{\text{convex}}$ , which has the following description in terms of the exponent matrix  $\mathbf{E}$  of  $\mathcal{M}$ :

$$(3.2.2) \quad \mathcal{M}^{\text{convex}} = \{ \mathbf{E} \mathbf{s} : \mathbf{s} \in \mathbb{R}_{\geq 0}^n \text{ and } |\mathbf{s}| = 1 \}.$$

**Definition 3.9.** We call  $\mathbf{N} := \mathcal{M}^{\text{convex}} + \mathbb{R}_{\geq 0}^m$  the *Newton polyhedron* of  $\mathcal{M}$ .

Note that  $\mathbf{N}$  may also be described as the convex hull of  $\{ \mathbf{v} : \mathbf{x}^{\mathbf{v}} \in (\mathcal{M}) \}$ , where  $(\mathcal{M})$  denotes the monomial ideal generated by  $\mathcal{M}$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_m$  denote the standard basis for  $\mathbb{R}^m$ . As  $\mathbb{R}_{\geq 0}^m = \{ \mathbf{e}_1, \dots, \mathbf{e}_m \}^{\text{cone}}$ , it follows from Discussion 3.8 that both  $\mathbf{P}$  and  $\mathbf{N}$  are rational polyhedral sets. The following lemma gives some important conditions on the defining inequalities of  $\mathbf{N}$ .

**Lemma 3.10.** Let  $L = \beta_1 z_1 + \cdots + \beta_m z_m \in \mathbb{Q}[z_1, \dots, z_m]$  be a non-zero linear form, and suppose that  $H_L^\beta$  is a supporting halfspace of  $\mathbf{N}$ . Then  $\beta_1, \dots, \beta_m \geq 0$ . Furthermore,  $L^{-1}(\beta) \cap \mathbf{N}$  is bounded if and only if  $\beta_1, \dots, \beta_m > 0$ .

*Proof.* Fix  $\mathbf{v} \in L^{-1}(\beta) \cap \mathbf{N}$ . If  $\lambda > 0$ , then  $\mathbf{v} + \lambda \cdot \mathbf{e}_i \in \mathbf{N}$ , and so

$$(3.2.3) \quad \beta \leq L(\mathbf{v} + \lambda \cdot \mathbf{e}_i) = L(\mathbf{v}) + \lambda \cdot L(\mathbf{e}_i) = \beta + \lambda \cdot \beta_i.$$

As  $\lambda > 0$ , (3.2.3) implies  $\beta_i \geq 0$ . We now prove the contrapositive of the second assertion.

The face  $L^{-1}(\beta) \cap \mathbf{N}$  is unbounded if and only if some ray  $\{\mathbf{v} + \lambda \cdot \mathbf{e}_i : \lambda \in \mathbb{R}_{>0}\}$  is contained in  $L^{-1}(\beta) \cap \mathbf{N}$ . However, this happens if and only if we have equality in (3.2.3), which then shows that  $0 = \lambda \cdot \beta_i$ . As  $\lambda > 0$ , we conclude that  $\beta_i = 0$ .  $\square$

**Remark 3.11.** In what follows, “ $\longleftrightarrow$ ” will be used to denote bijective correspondence between sets. If  $\lambda > 0$ , it follows from the definitions that

$$(3.2.4) \quad \begin{aligned} \mathbf{P}_\lambda &\longleftrightarrow \frac{1}{\lambda} \cdot \mathbf{P}_\lambda = \left\{ \mathbf{s} \in \mathbb{R}_{\geq 0}^n : |\mathbf{s}| = 1 \text{ and } \mathbf{E}\mathbf{s} \preccurlyeq \frac{1}{\lambda} \cdot \mathbf{1}_m \right\} \\ &\longleftrightarrow \left\{ (\mathbf{s}, \mathbf{w}) \in \mathbb{R}_{\geq 0}^n \times \mathbb{R}_{\geq 0}^m : |\mathbf{s}| = 1 \text{ and } \mathbf{E}\mathbf{s} + \mathbf{w} = \frac{1}{\lambda} \cdot \mathbf{1}_m \right\}. \end{aligned}$$

It follows from (3.2.4) and (3.2.2) that  $\mathbf{P}_\lambda \neq \emptyset \iff \frac{1}{\lambda} \cdot \mathbf{1}_m \in \mathbf{N}$ . In particular,

$$(3.2.5) \quad \mathbf{fpt}_m(\mathcal{M}) = \max \{ |\mathbf{s}| : \mathbf{s} \in \mathbf{P} \} = \max \left\{ \lambda > 0 : \frac{1}{\lambda} \cdot \mathbf{1}_m \in \mathbf{N} \right\},$$

where the first equality in (3.2.5) holds by Proposition 3.7.

### 3.3. Newton polyhedra in diagonal position, and $F$ -pure thresholds.

**Definition 3.12.** We will use  $\alpha$  to denote the common values in (3.2.5).

**Notation 3.13.** By definition,  $\frac{1}{\alpha} \cdot \mathbf{1}_m$  is contained in  $\partial \mathbf{N}$ , the boundary of  $\mathbf{N}$ . By [Web94, Theorem 3.2.2],  $\partial \mathbf{N}$  is the union of the faces of  $\mathbf{N}$ , and we use  $\mathbf{\Lambda}$  to denote the unique minimal face, with respect to inclusion, containing  $\frac{1}{\alpha} \cdot \mathbf{1}_m$ . We reorder  $\mathcal{M}$  and choose  $1 \leq r \leq n$  so that  $\mathbf{a}_i \in \mathbf{\Lambda}$  if and only if  $1 \leq i \leq r$ . We use  $\mathcal{M}_\mathbf{\Lambda}$  to denote the set  $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}\} \subseteq \mathcal{M}$ , and  $\mathbf{e}_1, \dots, \mathbf{e}_n$  to denote the standard basis of  $\mathbb{R}^n$ .

Suppose that  $L = \beta_1 z_1 + \dots + \beta_m z_m \in \mathbb{Q}[z_1, \dots, z_m]$  is a non-zero linear form and  $\beta$  is a rational number such that  $H_L^\beta$  is a supporting halfspace of  $\mathbf{N}$  and  $\frac{1}{\alpha} \cdot \mathbf{1}_m$  is on the face determined by  $H_L^\beta$ . By Lemma 3.10,  $\beta_1, \dots, \beta_m \geq 0$  and are not all zero, and so  $\beta = L(\frac{1}{\alpha} \cdot \mathbf{1}_m) = \frac{1}{\alpha} \cdot (\beta_1 + \dots + \beta_m) > 0$ . Normalizing, we obtain the following.

**Definition 3.14.** There exists a unique linear form  $L_\mathbf{\Lambda} \in \mathbb{Q}[z_1, \dots, z_m]$  with non-negative coefficients such that  $H_L^1$  is a supporting halfspace of  $\mathbf{\Lambda}$  and  $L_\mathbf{\Lambda}^{-1}(1) \cap \mathbf{N} = \mathbf{\Lambda}$ . We call  $L_\mathbf{\Lambda}$  the *linear form determined by  $\mathbf{\Lambda}$* .

**Definition 3.15.** We say that  $\mathbf{N}$  is in *diagonal position* if  $\mathbf{\Lambda}$  is bounded. By Lemma 3.10,  $\mathbf{N}$  is in diagonal position if and only if every  $z_i$  appears in  $L_\mathbf{\Lambda}$  with positive coefficient.

This terminology is motivated by the fact that if  $\mathcal{M} = \{x_1^{d_1}, \dots, x_m^{d_m}\}$ , then  $\mathcal{M} = \mathcal{M}_\mathbf{\Lambda}$  defines a Newton polyhedron in diagonal position.

**Example 3.16.** The Newton polyhedra associated to the collection of monomials appearing in Example 3.5 and 3.6 are in diagonal position.

**Lemma 3.17.** If  $\mathbf{N}$  is in diagonal position, then  $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{R}_{\geq 0}^n$ . Then  $\mathbf{s} \in \mathbf{P}_{\max}$  if and only if  $s_{r+1} = \dots = s_n = 0$  and  $\mathbf{E}\mathbf{s} = \mathbf{1}_m$ .



*Proof.* If  $\mathbf{s} \in \mathbf{P}_{\max}$ , set  $\boldsymbol{\kappa} := \frac{1}{\alpha} \cdot \mathbf{s}$ . Note that  $\boldsymbol{\kappa} \succcurlyeq \mathbf{0}$ ,  $|\boldsymbol{\kappa}| = 1$ , and  $\mathbf{E}\boldsymbol{\kappa} \preccurlyeq \frac{1}{\alpha} \cdot \mathbf{1}_m$ . By hypothesis,  $L_{\Lambda}$  has positive coefficients, and so preserves inequalities, and as  $L_{\Lambda} \equiv 1$  on  $\Lambda$ ,

(3.3.1)

$$\begin{aligned} 1 = L_{\Lambda} \left( \frac{1}{\alpha} \cdot \mathbf{1}_m \right) &\geq L_{\Lambda}(\mathbf{E}\boldsymbol{\kappa}) = L_{\Lambda} \left( \sum_{i=1}^n \kappa_i \cdot \mathbf{a}_i \right) = \sum_{i=1}^r \kappa_i \cdot L_{\Lambda}(\mathbf{a}_i) + \sum_{i=r+1}^n \kappa_i \cdot L_{\Lambda}(\mathbf{a}_i) \\ &= \sum_{i=1}^r \kappa_i + \sum_{i=r+1}^n \kappa_i \cdot L_{\Lambda}(\mathbf{a}_i). \end{aligned}$$

Substituting the equality  $|\boldsymbol{\kappa}| = 1$  into the initial term in (3.3.1), we see that

(3.3.2) 
$$\sum_{i=r+1}^n \kappa_i \geq \sum_{i=r+1}^n \kappa_i \cdot L_{\Lambda}(\mathbf{a}_i).$$

By definition,  $L_{\Lambda}(\mathbf{a}_i) > 1$  for  $r+1 \leq i \leq n$ , and so (3.3.2) is possible if and only if  $\kappa_i$  (and hence  $s_i$ ) is zero for  $r+1 \leq i \leq n$ . Furthermore, as the coefficients of  $L_{\Lambda}$  are positive, if any of the inequalities in  $\mathbf{E}\boldsymbol{\kappa} \preccurlyeq \frac{1}{\alpha} \cdot \mathbf{1}_m$  were strict, then the inequality in (3.3.1) would be strict as well, which would imply that  $1 > \sum_{i=1}^r \kappa_i = |\boldsymbol{\kappa}| = 1$ , a contradiction. We conclude that  $\mathbf{E}\boldsymbol{\kappa} = \frac{1}{\alpha} \cdot \mathbf{1}_m$ , and thus  $\mathbf{E}\mathbf{s} = \mathbf{1}_m$ .

Next, suppose that  $\mathbf{s} = \sum_{i=1}^r s_i \cdot \mathbf{e}_i \succcurlyeq \mathbf{0}$  and  $\mathbf{E}\mathbf{s} = 1$ . Thus,  $\sum_{i=1}^r s_i \cdot \mathbf{a}_i = \mathbf{E}\mathbf{s} = \mathbf{1}_m$ , and so  $|\mathbf{s}| = \sum_{i=1}^r s_i = \sum_{i=1}^r s_i \cdot L_{\Lambda}(\mathbf{a}_i) = L_{\Lambda}(\sum_{i=1}^r s_i \cdot \mathbf{a}_i) = L_{\Lambda}(\mathbf{1}_m) = \alpha \cdot L_{\Lambda}(\frac{1}{\alpha} \cdot \mathbf{1}_m) = \alpha$ .  $\square$

**Corollary 3.18.** There exists a maximal point  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$  with  $(p^e - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  if and only if  $(p^e - 1) \cdot \alpha \in \mathbb{N}$  and  $(\mathcal{M})^{(p^e-1)\cdot\alpha} \not\subseteq \mathfrak{m}^{[p^e]}$ . Furthermore, if either condition holds and  $\mathbf{N}$  is in diagonal position, then  $(\mathcal{M})^{(p^e-1)\cdot\alpha} \equiv (\mathcal{M}_{\Lambda})^{(p^e-1)\cdot\alpha} \equiv (x_1^{p^e-1} \cdots x_m^{p^e-1}) \bmod \mathfrak{m}^{[p^e]}$ .

*Proof.* Recall that  $(\mathcal{M})^N$  is generated by the monomials  $\mathbf{x}^{\mathbf{E}\mathbf{k}}$  with  $|\mathbf{k}| = N$ . If  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$  and  $(p^e - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$ , then  $(p^e - 1) \cdot \alpha = (p^e - 1) \cdot |\boldsymbol{\eta}| \in \mathbb{N}$ , and  $\mathbf{x}^{(p^e-1)\cdot\mathbf{E}\boldsymbol{\eta}}$  is in  $(\mathcal{M})^{(p^e-1)\cdot\alpha}$ , but not in  $\mathfrak{m}^{[p^e]}$ . Conversely, if  $(\mathcal{M})^{(p^e-1)\cdot\alpha} \not\subseteq \mathfrak{m}^{[p^e]}$ , there exists  $\mathbf{k} \in \mathbb{N}^n$  such that  $|\mathbf{k}| = (p^e - 1) \cdot \alpha$  and  $\mathbf{x}^{\mathbf{E}\mathbf{k}} \notin \mathfrak{m}^{[p^e]}$ , so that  $\frac{1}{p^e-1} \cdot \mathbf{k} \in \mathbf{P}_{\max}$ .

We have just seen that the monomials in  $(\mathcal{M})^{(p^e-1)\cdot\alpha}$  not contained in  $\mathfrak{m}^{[p^e]}$  are of the form  $\mathbf{x}^{\mathbf{E}\mathbf{k}}$  for some index  $\mathbf{k}$  satisfying  $\frac{1}{p^e-1} \cdot \mathbf{k} \in \mathbf{P}_{\max}$ . By Lemma 3.17,  $\mathbf{E}\mathbf{k} = (p^e - 1) \cdot \mathbf{1}_m$  and  $k_i = 0$  for  $r+1 \leq i \leq n$ , so that  $\mathbf{x}^{\mathbf{E}\mathbf{k}} = \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m}$  is in  $(\mathcal{M}_{\Lambda})^{(p^e-1)\cdot\alpha}$ .  $\square$

**Definition 3.19.** If  $g = \sum_{\mathbf{I}} \mathbf{u}_{\mathbf{I}} \cdot \mathbf{x}^{\mathbf{I}}$  is a polynomial over a field, we will use  $\text{Supp}(g)$  to denote  $\{\mathbf{x}^{\mathbf{I}} : \mathbf{u}_{\mathbf{I}} \neq 0\}$ , the set of *supporting monomials* of  $g$ . If  $\sigma \subseteq \text{Supp } g$ , we will use  $g_{\sigma}$  to denote the polynomial  $\sum_{\mathbf{x}^{\mathbf{I}} \in \sigma} \mathbf{u}_{\mathbf{I}} \cdot \mathbf{x}^{\mathbf{I}}$ , so that  $\text{Supp}(g_{\sigma}) = \sigma$ .

**Proposition 3.20.** Let  $f = \sum_{i=1}^n u_i \mathbf{x}^{\mathbf{a}_i}$  be a polynomial over a field  $\mathbb{L}$  of prime characteristic  $p$  with  $\text{Supp}(f) = \mathcal{M}$ . If  $\mathbf{N}$  is in diagonal position and  $(p^e - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  for some  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ , there exists a non-zero polynomial  $\Theta_e \in \mathbb{Z}[t_1, \dots, t_r]$  satisfying the following conditions:

- (1)  $f^{(p^e-1)\cdot\alpha} \equiv (f_{\Lambda})^{(p^e-1)\cdot\alpha} \equiv \Theta_e(u_1, \dots, u_r) \cdot x_1^{p^e-1} \cdots x_m^{p^e-1} \bmod \mathfrak{m}^{[p^e]}$ .
- (2)  $\Theta_{ed}(u_1, \dots, u_r) = \Theta_e(u_1, \dots, u_r)^{\frac{p^{ed}-1}{p^e-1}}$  for every  $d \geq 1$ .
- (3) If  $e = 1$  and  $\alpha \leq 1$ , then  $\Theta_1$  has non-zero image in  $\mathbb{F}_p[t_1, \dots, t_r]$ .

*Proof.* By definition,  $f_{\Lambda} = \sum_{i=1}^r u_i \cdot \mathbf{x}^{a_i}$ . By Corollary 3.18,  $f^{(p^e-1)\cdot\alpha} \equiv (f_{\Lambda})^{(p^e-1)\cdot\alpha} \bmod \mathfrak{m}^{[p^e]}$ , and is a multiple of  $\mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m}$  modulo  $\mathfrak{m}^{[p^e]}$ . The multinomial theorem then shows that  $f^{\Lambda}$  is  $\Theta_e(u_1, \dots, u_r) \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m} \bmod \mathfrak{m}^{[p^e]}$ , where

$$(3.3.3) \quad \Theta_e(t_1, \dots, t_r) := \sum_{\substack{\kappa \in \mathbb{N}^r, |\kappa| = (p^e-1)\cdot\alpha \\ \mathbf{E}\kappa = (p^e-1)\cdot\mathbf{1}_m}} \binom{(p^e-1)\cdot\alpha}{\kappa} t_1^{\kappa_1} \cdots t_r^{\kappa_r} \in \mathbb{Z}[t_1, \dots, t_r].$$

By assumption, there exists  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in \mathbf{P}_{\max}$  with  $(p^e-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$ . By Lemma 3.17,  $\eta_{r+1} = \dots = \eta_n = 0$  and  $\mathbf{E}\boldsymbol{\eta} = \mathbf{1}_m$ . Thus, the index  $\kappa = (p^e-1) \cdot \boldsymbol{\eta}$  corresponds to a non-zero summand in (3.3.3), and so  $\Theta_e \neq 0$ .

Next, set  $\gamma_d := \frac{p^{ed}-1}{p^e-1}$ . As  $f^{(p^e-1)\cdot\alpha} = \Theta_e(u_1, \dots, u_r) \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m} + g$  for some  $g \in \mathfrak{m}^{[p^e]}$ ,

$$(3.3.4) \quad f^{(p^e-1)\cdot\alpha} = (f^{(p^e-1)\cdot\alpha})^{\gamma_d} = \sum_{a+b=\gamma_d} \binom{\gamma_d}{a, b} (\Theta_e(u_1, \dots, u_r) \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m})^a g^b.$$

Let  $a, b$  be such that  $a+b = \gamma_d$  and consider the base  $p$  expansions  $a = \sum_{w=1}^{(d-1)e} a_w \cdot p^w$  and  $b = \sum_{w=1}^{(d-1)e} b_w \cdot p^w$ . As  $\gamma_d = 1 + p^e + \dots + p^{(d-2)e} + p^{(d-1)e}$  is the base  $p$  expansion of  $\gamma_d$ , it follows from Lemma 1.8 that, modulo  $p$ ,

$$(3.3.5) \quad \binom{\gamma_d}{a, b} \neq 0 \iff a_{e\ell} + b_{e\ell} = 1 \text{ for } 0 \leq \ell \leq d-1 \text{ and } a_w + b_w = 0 \text{ otherwise.}$$

Choose  $a$  and  $b$  with  $b \neq 0$  such that  $a+b = \gamma_d$  and  $\binom{\gamma_d}{a, b} \neq 0 \bmod p$ . As  $b \neq 0$ , it follows from (3.3.5) that  $L = \max \{ \ell : b_{e\ell} \neq 0 \}$  is well defined, and (3.3.5) again shows that

$$a \geq p^{e(L+1)} + \dots + p^{e(d-1)} \text{ and } b \geq p^{eL}.$$

As  $g \in \mathfrak{m}^{[p^e]}$ , it follows that  $g^b \in \mathfrak{m}^{[p^{e(L+1)}]}$ , and so

$$(3.3.6) \quad \begin{aligned} (\mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m})^a \cdot g^b &= (x_1 \cdots x_m)^{(p^e-1)a} \cdot g^b \\ &\in (x_1 \cdots x_m)^{(p^e-1)(p^{e(L+1)} + \dots + p^{e(d-1)})} \cdot \mathfrak{m}^{[p^{e(L+1)}]} \\ &= (x_1 \cdots x_m)^{p^{ed} - p^{e(L+1)}} \cdot \mathfrak{m}^{[p^{e(L+1)}]} \subseteq \mathfrak{m}^{[p^{ed}]}. \end{aligned}$$

Thus, substituting the conclusion from (3.3.6) into (3.3.4), we see that the only non-zero summand of  $f^{(p^e-1)\cdot\alpha}$  not contained in  $\mathfrak{m}^{[p^{ed}]}$  corresponds to the indices  $b=0, a=\gamma_d$ , i.e.  $f^{(p^e-1)\cdot\alpha} \equiv (\Theta_e(u_1, \dots, u_r) \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m})^{\gamma_d} \equiv \Theta_e(u_1, \dots, u_r)^{\gamma_d} \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m} \bmod \mathfrak{m}^{[p^{ed}]}$ .

For the last point, note that if  $\alpha \leq 1$ , then  $(p-1)\alpha < p$ , and hence the binomial coefficients in (3.3.3) are non-zero modulo  $p$ .  $\square$

Let  $f$  be as in Proposition 3.20, and suppose that  $\mathbf{N}$  is in diagonal position,  $(p-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  for some  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ , and  $\alpha \leq 1$ . By Proposition 3.20, the reduction of  $\Theta_1$  modulo  $p$  defines a non-empty closed set  $Z \subseteq \mathbb{A}_{\mathbb{F}}^n$  satisfying the following condition: If the coefficients of  $f$  are not in  $Z$ , then  $f^{(p-1)\cdot\alpha} \notin \mathfrak{m}^{[p]}$ . By Lemma 2.5, we conclude that  $\mathbf{fpt}_m(f) \geq \alpha = \mathbf{fpt}_m(\mathcal{M})$ , and it follows from Remark 2.7 that  $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(\mathcal{M})$ . We now show that the condition that  $\mathbf{N}$  be in diagonal position is not necessary to reach this conclusion.

**Proposition 3.21.** Let  $f$  be as in Proposition 3.20. If  $\mathbf{fpt}_m(\mathcal{M}) \leq 1$  and  $(p-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  for some  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ , there exists a non-empty closed set  $Z \subseteq \mathbb{A}_K^n$  such that  $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(\mathcal{M})$  whenever  $(u_1, \dots, u_n)$ , the coefficients of  $f$ , are not in  $Z$ .

**Remark 3.22.** The condition that  $(p-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  in Proposition 3.21 is necessary. Indeed, suppose that  $\mathcal{M} = \{x_1^n, \dots, x_n^n\}$ . We have seen in Example 3.5 that  $\mathbf{P}_{\max} = \{(\frac{1}{n}, \dots, \frac{1}{n})\}$ , so that  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) = \frac{1}{n} + \dots + \frac{1}{n} = 1$  by Proposition 3.7. However, it is shown in [Her10c] that if  $p \not\equiv 1 \pmod n$ , then  $\mathbf{fpt}_{\mathbf{m}}(f) < \mathbf{fpt}_{\mathbf{m}}(\mathcal{M})$  for every polynomial  $f$  with  $\text{Supp}(f) = \mathcal{M}$ .

*Proof of Proposition 3.21.* As explained in the paragraph immediately preceding the statement of Proposition 3.21, it suffices to define a non-empty closed set  $Z \subseteq \mathbb{A}_K^n$  such that  $f^{(p-1) \cdot \alpha} \notin \mathbf{m}^{[p]}$  whenever  $(u_1, \dots, u_n) \notin Z$ . Consider the polynomial

$$(3.3.7) \quad \Theta_{\boldsymbol{\eta}}(t_1, \dots, t_n) = \sum_{\substack{\mathbf{k} \in \mathbb{N}^n, |\mathbf{k}| = (p-1) \cdot \alpha \\ \mathbf{E}\mathbf{k} = (p-1)\mathbf{E}\boldsymbol{\eta}}} \binom{(p-1) \cdot \alpha}{k_1, \dots, k_n} t_1^{k_1} \dots t_n^{k_n} \in \mathbb{Z}[t_1, \dots, t_n].$$

As  $\alpha \leq 1$ , each of the binomial coefficients in (3.3.7) is not zero, and  $\mathbf{k} = (p-1) \cdot \boldsymbol{\eta}$  corresponds to a summand of  $\Theta_{\boldsymbol{\eta}}$ , so that  $\Theta_{\boldsymbol{\eta}} \not\equiv 0 \pmod p$ . The multinomial theorem shows that  $\Theta_{\boldsymbol{\eta}}(u_1, \dots, u_n)$  is the coefficient of the monomial  $\mathbf{x}^{(p-1)\mathbf{E}\boldsymbol{\eta}}$  (which is not in  $\mathbf{m}^{[p]}$  as  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ ) in  $f^{(p-1)\alpha}$ . Thus, we may take  $Z = \mathbb{V}(\Theta_{\boldsymbol{\eta}}) \subseteq \mathbb{A}_{\mathbb{L}}^n$ .  $\square$

Let  $f$  be a polynomial with  $\text{Supp}(f) = \mathcal{M}$ . By Remark 2.7,

$$(3.3.8) \quad \mathbf{fpt}_{\mathbf{m}}(f) \leq \min \{1, \mathbf{fpt}_{\mathbf{m}}(\mathcal{M})\}.$$

We have seen in Proposition 3.21 that if  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) \leq 1$  and  $(p-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$  for some  $\boldsymbol{\eta} \in \mathbf{P}_{\max}$ , then we have equality in (3.3.8) for a general choice of coefficients. The following theorem shows that if  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) > 1$  and  $f_{\Lambda}$  has an isolated singularity at the origin, then equality in (3.3.8) holds provided that  $p$  is large enough.

**Lemma 3.23.** [Fed83, Lemma 2.3] Let  $f$  be a polynomial over a field of characteristic  $p > 0$  with  $\text{Supp}(f) = \mathcal{M}$ . Suppose  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) > 1 > \mathbf{fpt}_{\mathbf{m}}(f)$  and that  $f_{\Lambda}$  has an isolated singularity at the origin, so that  $(x_1^N, \dots, x_m^N) \subseteq \left(\frac{\partial f_{\Lambda}}{\partial x_1}, \dots, \frac{\partial f_{\Lambda}}{\partial x_m}\right)$  for some  $N \geq 1$ . If  $p^e > N$ , then  $\nu_{f_{\Lambda}}(p^e) \geq (p^e - N) \cdot \mathbf{fpt}_{\mathbf{m}}(\mathcal{M})$ . In particular,  $p < \left(N \cdot \frac{\mathbf{fpt}_{\mathbf{m}}(\mathcal{M})}{\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) - 1}\right)^{1/e}$ .

**Remark 3.24.** Lemma 3.23 is a generalization of [Fed83, Lemma 2.3], which applies to *quasi-homogeneous* polynomials. One may verify that  $f$  is quasi-homogeneous in the sense of [Fed83] if and only if  $\mathbf{N}$  is in diagonal position and  $f = f_{\Lambda}$ .

**Remark 3.25.** If  $m = 2$ , the hypothesis that  $f_{\Lambda}$  have an isolated singularity at  $\mathbf{m}$  is superfluous, as it is implied by the assumption that  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) > 1$ . Indeed, we see from (3.2.5) that  $\mathbf{fpt}_{\mathbf{m}}(\mathcal{M}) > 1$  if and only if  $(1, 1) \in \mathbf{N}^{\circ}$ , the interior of  $\mathbf{N}$ . In this simplified setting, it is apparent that  $(1, 1) \in \mathbf{N}^{\circ}$  if and only if, after possibly reordering the variables,  $\Lambda = \{x_1, x_2^d\}$  for some  $d \geq 1$ . In this case, we see that  $\mathbf{N}$  is in diagonal position, and that  $f_{\Lambda}$  is a  $K^*$ -linear combination of  $x_1$  and  $x_2^d$ , and is thus regular at all points.

*Proof of Lemma 3.23.* As  $\mathbf{fpt}_{\mathbf{m}}(f) < 1$ , it follows from setting  $\lambda = 1$  in Lemma 2.5 that  $f^{p^e-1} \in \mathbf{m}^{[p^e]}$ . We claim that  $f_{\Lambda}^{p^e-1} \in \mathbf{m}^{[p^e]}$  as well. By definition, there exists  $g$  such that

$$(3.3.9) \quad f^{p^e-1} = f_{\Lambda}^{p^e-1} + g.$$

If  $f_{\Lambda}^{p^e-1} \notin \mathbf{m}^{[p^e]}$ , there exists a monomial  $\text{Supp}(f_{\Lambda}^{p^e-1})$  not contained in  $\mathbf{m}^{[p^e]}$ . By definition such a monomial is of the form  $\mathbf{x}^{\mathbf{E}\mathbf{v}}$  for some index  $\mathbf{v} = \sum_{i=1}^r v_i \cdot \mathbf{e}_i$ . As  $f^{p^e-1} \in \mathbf{m}^{[p^e]}$ , it follows from (3.3.9) that there exists a monomial  $\mathbf{x}^{\mathbf{E}\mathbf{w}} \in \text{Supp}(g)$  whose corresponding

summand cancels the one determined by  $\mathbf{x}^{\mathbf{E}\mathbf{v}}$ , so that  $|\mathbf{w}| = |\mathbf{v}|$ ,  $\mathbf{E}\mathbf{w} = \mathbf{E}\mathbf{v}$ . It follows by definition of  $g$  that  $\mathbf{w} = \sum_{i=1}^n w_i \cdot \mathbf{e}_i$ , with  $w_i \neq 0$  for some  $i > r$ . As  $L_\Lambda \equiv 1$  on  $\Lambda$ ,

$$\begin{aligned} |\mathbf{v}| &= \sum_{i=1}^r v_i = \sum_{i=1}^r v_i \cdot L_\Lambda(\mathbf{a}_i) = L_\Lambda(\mathbf{E}\mathbf{v}) = L_\Lambda(\mathbf{E}\mathbf{w}) = L_\Lambda\left(\sum_{i=1}^r w_i \cdot \mathbf{a}_i + \sum_{i=r+1}^n w_i \cdot \mathbf{a}_i\right) \\ (3.3.10) \quad &= \sum_{i=1}^r w_i + \sum_{i=r+1}^n w_i \cdot L(\mathbf{a}_i) > |\mathbf{w}| = |\mathbf{v}|. \end{aligned}$$

In obtaining the contradiction in (3.3.10), we used that  $w_i \neq 0$  for some  $i > r$  and that  $L_\Lambda(\mathbf{a}_i) > 1$  whenever  $i > r$ .

Next, let  $\nu_\Lambda(p^e) := \nu_{f_\Lambda}(p^e)$  be as in (2.0.4). We have just shown that  $f_\Lambda^{p^e-1} \in \mathfrak{m}^{[p^e]}$ , and by definition  $\nu_\Lambda(p^e) \leq p^e - 2$ . As  $f_\Lambda^{\nu_\Lambda(p^e)+1} \in \mathfrak{m}^{[p^e]}$ , taking derivatives (keeping in mind that  $\nu_\Lambda(p^e) \leq p^e - 2$  and that we are over  $\mathbb{F}_p$ ) shows that  $\frac{\partial f_\Lambda}{\partial x_i} \cdot f_\Lambda^{\nu_\Lambda(p^e)} \in \mathfrak{m}^{[p^e]}$  for  $1 \leq i \leq n$ . It follows from this, and our original hypotheses, that

$$f_\Lambda^{\nu_\Lambda(p^e)} \in \left( \mathfrak{m}^{[p^e]} : \frac{\partial f_\Lambda}{\partial x_1}, \dots, \frac{\partial f_\Lambda}{\partial x_m} \right) \subseteq (\mathfrak{m}^{[p^e]} : x_1^N, \dots, x_m^N) = (x_1 \cdots x_m)^{p^e-N}.$$

Thus, there exists  $\mathbf{k} = \sum_{i=1}^r k_i \mathbf{e}_i \in \mathbb{N}^r$  with  $|\mathbf{k}| = \nu_\Lambda(p^e)$  such that  $\mathbf{x}^{\mathbf{E}\mathbf{k}} \in \text{Supp}\left(f_\Lambda^{\nu_\Lambda(p^e)}\right)$  and  $\mathbf{E}\mathbf{k} \succcurlyeq (p^e - N) \cdot \mathbf{1}_m$ . Applying  $L_\Lambda$  to this inequality yields

$$\nu_\Lambda(p^e) = \sum_{i=1}^n k_i \cdot L_\Lambda(\mathbf{a}_i) = L_\Lambda(\mathbf{E}\mathbf{k}) \geq (p^e - N) \cdot L_\Lambda(\mathbf{1}_m) = (p^e - N) \cdot \alpha,$$

where we have again used that  $L_\Lambda(\mathbf{1}_m) = \alpha \cdot L_\Lambda\left(\frac{1}{\alpha} \cdot \mathbf{1}_m\right) = \alpha$ . A straightforward manipulation shows that  $\nu_\Lambda(p^e) \geq (p^e - N) \cdot \alpha$  if and only if  $\alpha N \geq p^e \alpha - \nu_\Lambda(p^e)$ , and so

$$(3.3.11) \quad \alpha N \geq p^e \alpha - \nu_\Lambda(p^e) = p^e(\alpha - 1) + p^e - \nu_\Lambda(p^e) > p^e(\alpha - 1),$$

where we have used that  $\nu_\Lambda(p^e) \leq p^e - 1$ . As we are assuming that  $\alpha > 1$ , (3.3.11) shows that  $p^e < \left(\frac{\alpha}{\alpha-1} \cdot N\right)$ , and so we are done.  $\square$

### 3.4. The unique maximal point condition.

**Definition 3.26.** We say that  $\mathbf{P}$  contains a unique maximal point if  $\#\mathbf{P}_{\max} = 1$ .

**Remark 3.27.** By definition,  $\mathbf{P}_{\max}$  is a face of  $\mathbf{P}$ . Thus, if  $\mathbf{P}_{\max} = \{\boldsymbol{\eta}\}$ , then  $\boldsymbol{\eta}$  must be a vertex of  $\mathbf{P}$ , and as such has rational coordinates. Additionally, if  $H = \{\mathbf{s} \in \mathbb{R}^n : L(\mathbf{s}) \leq 1\}$  is a (lower) halfspace defined by a linear form  $L$  with rational coefficients, then every vertex of the rational polyhedral set  $\mathbf{P} \cap H$  must also have rational coordinates.

**Example 3.28.** The polytope from Example 3.5 always has a unique maximal point. If  $\mathbf{P}$  is the polytope from Example 3.6, we see that  $\mathbf{P}$  contains a unique maximal point if and only if  $|\mathbf{v}_3| = \frac{1}{a} + \frac{1}{b} \neq \frac{1}{c} = |\mathbf{v}_4|$ , in which case  $\mathbf{P}_{\max} = \{\mathbf{v}_3\}$  or  $\{\mathbf{v}_4\}$ . If  $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$ , then  $\mathbf{P}_{\max}$  is the edge connecting  $\mathbf{v}_3$  and  $\mathbf{v}_4$ .

**Remark 3.29.** Recall that a set  $\{\mathbf{b}_1, \dots, \mathbf{b}_r\}$  is said to be *affinely independent* if  $\mathbf{0}$  is the unique solution to the system of equations  $\sum_{i=1}^r k_i \cdot \mathbf{b}_i = \mathbf{0}$  and  $\sum_{i=1}^r k_i = 0$ . If  $\mathbf{N}$  is in diagonal position, it follows from Proposition 3.17 that  $\mathbf{P}$  contains a unique maximal point if

and only if there is a unique solution to the system  $\mathbf{s} \succcurlyeq 0$ ,  $\mathbf{E}\mathbf{s} = \mathbf{1}_m$ , and  $s_{r+1} = \dots = s_n = 0$ . In this case, we see that  $\#P_{\max} = 1$  if the exponents of  $\mathcal{M}_\Lambda$  are affinely independent.

**Lemma 3.30.** Suppose  $P$  has a unique maximal point  $\boldsymbol{\eta} \in P$ .

- (1) If  $|\mathbf{s}| = |\langle \boldsymbol{\eta} \rangle_e|$  and  $\mathbf{E}\mathbf{s} = \mathbf{E}\langle \boldsymbol{\eta} \rangle_e$  for some  $\mathbf{s} \succcurlyeq 0$ , then  $\mathbf{s} = \langle \boldsymbol{\eta} \rangle_e$ .
- (2) If  $|\mathbf{s}| = |\boldsymbol{\nu}|$ ,  $\mathbf{E}\mathbf{s} = \mathbf{E}\boldsymbol{\nu}$ , and  $\boldsymbol{\nu} \preccurlyeq \langle \boldsymbol{\eta} \rangle_e$  for some  $\boldsymbol{\nu}, \mathbf{s} \succcurlyeq 0$ , then  $\mathbf{s} = \boldsymbol{\nu}$ .

*Proof.* To prove the first statement, let  $\boldsymbol{\eta}' := \mathbf{s} + \boldsymbol{\eta} - \langle \boldsymbol{\eta} \rangle_e$ . By hypothesis,  $\boldsymbol{\eta}' \succcurlyeq \mathbf{s} \succcurlyeq 0$ ,  $\mathbf{E}\boldsymbol{\eta}' = \mathbf{E}\mathbf{s} + \mathbf{E}\boldsymbol{\eta} - \mathbf{E}\langle \boldsymbol{\eta} \rangle_e = \mathbf{E}\boldsymbol{\eta}$ , and  $|\boldsymbol{\eta}'| = |\mathbf{s}| + |\boldsymbol{\eta}| - |\langle \boldsymbol{\eta} \rangle_e| = |\boldsymbol{\eta}|$ , which shows that  $\boldsymbol{\eta}'$  is a maximal point of  $P$ . Thus  $\boldsymbol{\eta}' = \boldsymbol{\eta}$ , and  $\mathbf{s} = \langle \boldsymbol{\eta} \rangle_e$ .

For the second statement, let  $\mathbf{s}' := \mathbf{s} + \langle \boldsymbol{\eta} \rangle_e - \boldsymbol{\nu}$ . By hypothesis,  $\mathbf{s}' \succcurlyeq 0$ ,  $|\mathbf{s}'| = |\langle \boldsymbol{\eta} \rangle_e|$ , and  $\mathbf{E}\mathbf{s}' = \mathbf{E}\langle \boldsymbol{\eta} \rangle_e$ . The first statement, applied to  $\mathbf{s}'$ , shows that  $\mathbf{s}' = \langle \boldsymbol{\eta} \rangle_e$ , and thus  $\mathbf{s} = \boldsymbol{\nu}$ .  $\square$

**Corollary 3.31.** Suppose  $P$  has a unique maximal point  $\boldsymbol{\eta} \in P$ , and let  $f$  be a polynomial with  $\text{Supp}(f) = \mathcal{M}$  and coefficients  $u_1, \dots, u_n$ .

- (1) The coefficient of the monomial  $\mathbf{x}^{p^e \mathbf{E}\langle \boldsymbol{\eta} \rangle_e}$  in  $f^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$  is  $\binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \mathbf{u}^{p^e \langle \boldsymbol{\eta} \rangle_e}$ .
- (2) If  $\boldsymbol{\nu} \in \frac{1}{p^e} \cdot \mathbb{N}^n$  is an index such that  $\boldsymbol{\nu} \preccurlyeq \langle \boldsymbol{\eta} \rangle_e$ , then the coefficient of the monomial  $\mathbf{x}^{p^e \mathbf{E}\boldsymbol{\nu}}$  in  $f^{p^e |\boldsymbol{\nu}|}$  is  $\binom{p^e |\boldsymbol{\nu}|}{p^e \boldsymbol{\nu}} \mathbf{u}^{p^e \boldsymbol{\nu}}$ .

**Corollary 3.32.** Suppose that  $N$  is in diagonal position,  $P$  contains a unique maximal point  $\boldsymbol{\eta}$ , and  $p$  does not divide any of the denominators in  $\boldsymbol{\eta}$ . If  $f$  is a polynomial with  $\text{Supp}(f) = \mathcal{M}$  and coefficients  $u_1, \dots, u_n$ , then there exist infinitely many  $e$  such that  $(p^e - 1) \cdot \boldsymbol{\eta} = p^e \langle \boldsymbol{\eta} \rangle_e$  and  $f^{(p^e - 1) \cdot \alpha} = f^{p^e |\langle \boldsymbol{\eta} \rangle_e|} \equiv \binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \mathbf{u}^{p^e \langle \boldsymbol{\eta} \rangle_e} \cdot \mathbf{x}^{(p^e - 1) \cdot \mathbf{1}_m} \pmod{\mathfrak{m}^{[p^e]}}$ .

*Proof.* If  $p$  does not divide  $d$ , then  $p$  has finite order in the multiplicative group  $(\mathbb{Z}/d\mathbb{Z})^\times$ . It follows that there exists an  $e$  (and hence, infinitely many  $e$ ) with  $(p^e - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$ , and Lemma 1.4 implies that  $p^e \langle \boldsymbol{\eta} \rangle_e = (p^e - 1) \cdot \boldsymbol{\eta}$  for such  $e$ . Applying Proposition 3.20 shows that  $f^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$  is congruent to  $\gamma \cdot \mathbf{x}^{(p^e - 1) \cdot \mathbf{1}_m}$  modulo  $\mathfrak{m}^{[p^e]}$ ; that  $\gamma = \binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \mathbf{u}^{p^e \langle \boldsymbol{\eta} \rangle_e}$  may then be deduced from (3.3.3) or Corollary 3.31.  $\square$

#### 4. PROOF OF THE MAIN THEOREM

**Notation 4.1.**  $\mathcal{M}$ ,  $P$ , and  $N$  will continue to be as in the previous section. Furthermore,  $\mathbb{L}$  will denote an  $F$ -finite field of characteristic  $p$ , and  $f$  will denote a polynomial over  $\mathbb{L}$  with  $\text{Supp}(f) = \mathcal{M}$ .

**Theorem 4.2.** Suppose that  $P$  contains a unique maximal point  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n) \in P$ , and let  $L = \sup\{e : \eta_1^{(d)} + \dots + \eta_n^{(d)} \leq p - 1 \text{ for } 0 \leq d \leq e\}$ , where  $\eta_i^{(d)}$  is the  $d^{\text{th}}$  digit of  $\eta_i$ .

- (1) If  $L = \infty$ , then  $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(\mathcal{M})$ . The converse holds if  $N$  is in diagonal position and  $p$  does not divide any of the denominators of  $\boldsymbol{\eta}$ .
- (2) If  $L < \infty$ , then  $\mathbf{fpt}_m(f) \geq \langle \eta_1 \rangle_L + \dots + \langle \eta_n \rangle_L + \frac{1}{p^L}$ .

*Proof of Theorem 4.2.* Write  $f = \sum_{i=1}^n u_i \mathbf{x}^{a_i}$  as a  $\mathbb{L}^*$ -linear combination of the monomials in  $\mathcal{M}$ . We first prove (1), and thus assume that entries of  $\boldsymbol{\eta}$  add without carrying (in base  $p$ ). Corollary 3.31 give that, after gathering terms,

$$(4.0.1) \quad \binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \mathbf{u}^{p^e \langle \boldsymbol{\eta} \rangle_e} \mathbf{x}^{p^e \mathbf{E}\langle \boldsymbol{\eta} \rangle_e}$$

appears as a summand of  $f^{p^e|\langle \boldsymbol{\eta} \rangle_e|}$ . By definition, each  $u_i \in \mathbb{L}^*$ , so  $\mathbf{u}^{p^e\langle \boldsymbol{\eta} \rangle_e} \neq 0$ , while the assumption on the entries of  $\boldsymbol{\eta}$  implies the integers  $p^e \langle \eta_1 \rangle_e, \dots, p^e \langle \eta_n \rangle_e$  add without carrying as well, and applying Lemma 1.8 then shows that  $\binom{p^e|\langle \boldsymbol{\eta} \rangle_e|}{p^e\langle \boldsymbol{\eta} \rangle_e} \not\equiv 0 \pmod{p}$ . Finally,  $\langle \boldsymbol{\eta} \rangle_e \preccurlyeq \boldsymbol{\eta}$ , and so every entry of  $p^e \mathbf{E} \langle \boldsymbol{\eta} \rangle_e$  is less than  $p^e - 1$ . We see then that the monomial in (4.0.1) is in  $\text{Supp}(f^{p^e|\langle \boldsymbol{\eta} \rangle_e|})$ , but not in  $\mathfrak{m}^{[p^e]}$ . Thus,  $f^{p^e|\langle \boldsymbol{\eta} \rangle_e|} \notin \mathfrak{m}^{[p^e]}$ , and so  $\frac{\nu_f(p^e)}{p^e} \geq |\langle \boldsymbol{\eta} \rangle_e|$ . Letting  $e \rightarrow \infty$  shows that  $\mathbf{fpt}_m(f) \geq |\boldsymbol{\eta}|$ , while the opposite inequality holds by Remark 2.7.

We now address the second statement of the first point. Suppose then, by means of contradiction, that  $L < \infty$ , yet  $\mathbf{fpt}_m(f) = \mathbf{fpt}_m(\mathcal{M}) = \alpha$ . By Corollary 3.32, there exists  $e > L$  such that  $(p^e - 1) \cdot \boldsymbol{\eta} \in \mathbb{N}$  and

$$(4.0.2) \quad f^{(p^e-1)\cdot\alpha} = f^{p^e\langle \boldsymbol{\eta} \rangle_e} \equiv \binom{p^e|\langle \boldsymbol{\eta} \rangle_e|}{p^e\langle \boldsymbol{\eta} \rangle_e} \mathbf{u}^{p^e\langle \boldsymbol{\eta} \rangle_e} \cdot \mathbf{x}^{(p^e-1)\cdot\mathbf{1}_m} \pmod{\mathfrak{m}^{[p^e]}}.$$

As  $e > L$ , it follows from the definition of  $L$  that the entries of  $p^e \langle \boldsymbol{\eta} \rangle_e$  do not add without carrying, so that  $\binom{p^e|\langle \boldsymbol{\eta} \rangle_e|}{p^e\langle \boldsymbol{\eta} \rangle_e} \equiv 0 \pmod{p}$  by Lemma 1.8. It then follows from (4.0.2) that

$$(4.0.3) \quad f^{(p^e-1)\cdot\alpha} \in \mathfrak{m}^{[p^e]}.$$

However, the assumption that  $\mathbf{fpt}_m(f) = \alpha$  implies that  $\alpha \in (0, 1]$ , and as  $(p^e - 1) \cdot \alpha \in \mathbb{N}$ , Lemma 2.5 implies that  $f^{(p^e-1)\cdot\alpha} \notin \mathfrak{m}^{[p^e]}$ , contradicting (4.0.2).

We now address the second point. Recall that, by convention,  $\eta_i^{(0)} = 0$ , so that  $L \geq 0$ . Furthermore, it follows from the definition of  $L$  that  $\eta_1^{(L+1)} + \dots + \eta_n^{(L+1)} \geq p$ . After subtracting off from each  $\eta_i^{(L+1)}$  as necessary, we obtain integers  $\delta_1, \dots, \delta_n$  such that

$$(4.0.4) \quad \delta_1 + \dots + \delta_n = p - 1 \text{ and } 0 \leq \delta_i \leq \eta_i^{(L+1)},$$

with the second inequality in (4.0.4) being strict for at least one index. Without loss of generality, we assume that  $\delta_1 < \eta_1^{(L+1)}$ . For  $e \geq L + 2$ , set

$$(4.0.5) \quad \boldsymbol{\lambda}(e) = \langle \boldsymbol{\eta} \rangle_L + \left( \frac{\delta_1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \dots + \frac{p-1}{p^e}, \frac{\delta_2}{p^{L+1}}, \dots, \frac{\delta_n}{p^{L+1}} \right).$$

We now summarize some important properties of  $\boldsymbol{\lambda}(e)$ : By construction,  $\boldsymbol{\lambda}(e) \in \frac{1}{p^e} \cdot \mathbb{N}^n$ , and the definition of the  $\delta_i$  (along with our assumption that  $\delta_i < \eta_i^{(L+1)}$ ) shows that  $\boldsymbol{\lambda}(e) \prec \langle \boldsymbol{\eta} \rangle_e$ . Additionally, as  $\delta_1 + \dots + \delta_n = p - 1$ , it follows from the definition of  $L$  that the entries of  $p^e \cdot \boldsymbol{\lambda}(e)$  add without carrying (in base  $p$ ). Finally, we have

$$|\boldsymbol{\lambda}(e)| = |\langle \boldsymbol{\eta} \rangle_L| + \frac{1}{p^{L+1}} \cdot \left( \sum_{i=1}^n \delta_i \right) + \frac{p-1}{p^{L+2}} + \dots + \frac{p-1}{p^e} = |\langle \boldsymbol{\eta} \rangle_L| + \frac{p-1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \dots + \frac{p-1}{p^e}.$$

These properties imply that  $\binom{p^e|\boldsymbol{\lambda}(e)|}{p^e\boldsymbol{\lambda}(e)} \mathbf{u}^{p^e\boldsymbol{\lambda}(e)} \mathbf{x}^{p^e\mathbf{E}\boldsymbol{\lambda}(e)}$  is a non-zero summand of  $f^{p^e|\boldsymbol{\lambda}(e)|}$  and is not contained in  $\mathfrak{m}^{[p^e]}$ , so  $f^{p^e|\boldsymbol{\lambda}(e)|} \notin \mathfrak{m}^{[p^e]}$ . Thus,  $\frac{\nu_f(p^e)}{p^e} \geq |\boldsymbol{\lambda}(e)| = |\langle \boldsymbol{\eta} \rangle_L| + \sum_{d=L+1}^e \frac{p-1}{p^d}$ , and the assertion follows by letting  $e \rightarrow \infty$ .  $\square$

**Remark 4.3.** The estimates given in Theorem 4.2 can be used to calculate  $\mathbf{fpt}_m(f)$  in any characteristic whenever  $f$  is either a diagonal or binomial hypersurface [Her10c, Her10a].

## 5. LOG CANONICAL SINGULARITIES AND DENSE $F$ -PURE TYPE

Throughout this section,  $S$  will denote the polynomial ring  $\mathbb{C}[x_1, \dots, x_m]$ , and  $\mathcal{M}$  will continue to denote  $\{\mathbf{x}^{a_1}, \dots, \mathbf{x}^{a_n}\}$ , a collection of  $n$  monomials in the variables  $x_1, \dots, x_m$ .



### 5.1. The log canonical threshold of a polynomial.

**Definition 5.1.** Let  $f \in S$  be a non-zero polynomial such that  $f(\mathbf{0}) = 0$ . Then,

$$\mathbf{lct}_0(f) := \sup \left\{ \lambda > 0 : \frac{1}{|f|^{2\lambda}} \text{ is locally integrable at } \mathbf{0} \right\}$$

exists, and is called the *log canonical threshold of  $f$  at  $\mathbf{0}$* .

The invariant  $\mathbf{lct}_0(f)$  can be thought of as measuring the singularities of  $f$  near  $\mathbf{0}$ , with smaller values corresponding to “worse” singularities. One important property of these invariants is that  $\mathbf{lct}_0(f) \in (0, 1] \cap \mathbb{Q}$ . Though this is not obvious from Definition 5.1, it follows immediately from an alternate characterization of  $\mathbf{lct}_0(f)$  in terms of (log) resolution of singularities. For this alternate definition, see the survey [EM06].

By imposing similar local integrability conditions on the members of an ideal  $\mathfrak{a}$  of  $S$ , one may define  $\mathbf{lct}_0(\mathfrak{a})$ , the log canonical threshold at  $\mathbf{0}$  of  $\mathfrak{a}$ , as follows: If  $\mathfrak{a} = (f_1, \dots, f_d)$ , then

$$(5.1.1) \quad \mathbf{lct}_0(\mathfrak{a}) = \sup \left\{ \lambda > 0 : (|f_1|^2 + \dots + |f_d|^2)^{-\lambda} \text{ is locally integrable at } \mathbf{0} \right\}.$$

Using resolution of singularities, one sees that  $\mathbf{lct}_0(\mathfrak{a})$  is well defined and rational, and apparently,  $\mathbf{lct}_0(f) \leq \mathbf{lct}_0(\mathfrak{a})$  if  $f \in \mathfrak{a}$ . We gather these facts, along with a familiar formula for  $\mathbf{fpt}_m(\mathcal{M})$  which may be deduced from [How01, Example 5] below.

**Proposition 5.2.** [How01] Let  $f$  be a polynomial over  $\mathbb{C}$  with  $\text{Supp}(f) = \mathcal{M}$ . Then,  $\mathbf{lct}_0(f)$  and  $\mathbf{lct}_0(\mathfrak{a})$  are rational numbers, and  $\mathbf{lct}_0(f) \leq \min \{1, \mathbf{lct}_0(\mathcal{M})\}$ . We also have that  $\mathbf{lct}_0(\mathcal{M}) = \max \left\{ \lambda > 0 : \frac{1}{\lambda} \cdot \mathbf{1}_m \in \mathbf{N} \right\}$ .

Comparing the statements in Proposition 5.2 with those in Remark 2.7 shows that log canonical thresholds and  $F$ -pure threshold satisfy seemingly dual conditions. Furthermore, it follows from (3.2.5) and Proposition 5.2 that

$$(5.1.2) \quad \mathbf{fpt}_m(\mathcal{M}) = \max \{ |\mathbf{s}| : \mathbf{s} \in \mathbf{P} \} = \max \left\{ \lambda > 0 : \frac{1}{\lambda} \cdot \mathbf{1}_m \in \mathbf{N} \right\} = \mathbf{lct}_0(\mathcal{M}),$$

an observation (well-known among experts) that reveals the first of many deep connections between  $F$ -pure and log canonical thresholds. In order to precisely state this relationship, we review the process of *reduction to positive characteristic*.

**5.2. On reduction to positive characteristic.** If  $f \in \mathbb{Q}[x_1, \dots, x_m]$ , we may reduce the coefficients of  $f$  modulo  $p \gg 0$  to obtain a *family of positive characteristic models*  $\{f_p\}$  of  $f$  over the finite fields  $\mathbb{F}_p$  for all but finitely many  $p$ . Instead suppose that  $f \in \mathbb{C}[x_1, \dots, x_m]$  but does not have rational coefficients. Let  $A$  be a finitely-generated  $\mathbb{Z}$ -algebra containing the coefficients of  $f$ , so that  $f \in A[x_1, \dots, x_m]$ . For such an algebra,  $A/\mu$  is a finite field for every  $\mu \in \text{mSpec } A$ , and all but finitely many primes appear in the set  $\{\text{char } A/\mu : \mu \in \text{mSpec } A\}$ . Let  $f_\mu$  denote the image of  $f$  in  $(A/\mu)[x_1, \dots, x_m]$ . If  $f(\mathbf{0}) = 0$ , we may enlarge  $A$  (say, by adjoining the inverses of the coefficients of  $f$ ) so as to assume that  $\text{Supp}(f_\mu) = \text{Supp}(f)$  for every  $\mu \in \text{mSpec } A$ . We again call the set  $\{f_\mu : \mu \in \text{mSpec } A\}$  a family of positive characteristic models of  $f$ . In Corollary 5.4, we justify some these assertions, and our main tool will be the following variant of Noether Normalization.

**Lemma 5.3.** Let  $A$  be a finitely-generated algebra over a domain  $D$ . There exists non-zero element  $N \in D$  such that  $D_N \subseteq A_N$  factors as  $D_N \subseteq D_N[z_1, \dots, z_d] \subseteq A_N$ , where  $z_1, \dots, z_d$  are algebraically independent over  $D_N$ , and  $D_N[z_1, \dots, z_d] \subseteq A_N$  is finite.

If  $L = \text{Frac } D$  and  $R$  is the localization of  $A$  at the non-zero elements of  $D$ , then Lemma 5.3 can be obtained by applying the Noether Normalization theorem to the inclusion  $L \subseteq R$ . See [Hoc] for an alternate proof that does not rely on Normalization for algebras over a field.

**Corollary 5.4.** Every maximal ideal of a finitely-generated  $\mathbb{Z}$ -algebra  $A$  contains a prime  $p$ , and  $A/\mu$  is a finite field for every maximal ideal  $\mu \subseteq A$ . Furthermore, all but finitely many primes  $p$  are contained in a maximal ideal of  $A$ .

*Proof.* Let  $\mu \subseteq A$  be a maximal ideal. If  $\mu \cap \mathbb{Z} = 0$ , then  $A/\mu$  is also a finitely generated  $\mathbb{Z}$ -algebra. By Lemma 5.3,  $A/\mu = (A/\mu)_N$  is module finite over a polynomial ring with coefficients in  $\mathbb{Z}_N$  for some  $N$ , so that  $0 = \dim A/\mu \geq \dim \mathbb{Z}_N = 1$ , a contradiction. If  $p \in \mu$ , then  $A/\mu$  is finitely-generated over  $\mathbb{F}_p$ , and thus is module finite over a polynomial ring  $\mathbb{F}_p[z_1, \dots, z_d]$  by Lemma 5.3. As  $A/\mu$  is a field, dimension considerations force that  $d = 0$ , and thus  $A/\mu$  is finite over  $\mathbb{F}_p$ . Finally, consider a factorization  $\mathbb{Z}_N \subseteq \mathbb{Z}_N[z_1, \dots, z_d] \subseteq A_N$  as in Lemma 5.3. Every  $p$  not dividing  $N$  generates a prime ideal in the polynomial ring  $\mathbb{Z}_N[z_1, \dots, z_d]$ , and so by the Lying Over Theorem, there exists a prime (and hence maximal) ideal of  $A$  not containing  $N$  and lying over  $p$ .  $\square$

**Corollary 5.5.** Let  $A$  be a finitely-generated algebra over a domain  $D$ . Then, the inverse image of a dense set under the induced map  $\text{Spec } A \xrightarrow{\pi} \text{Spec } D$  is also dense.

*Proof.* Let  $\Gamma$  be dense in  $\text{Spec } D$ . It suffices to show that  $\text{Spec } A_f \cap \pi^{-1}(\Gamma)$  is non-empty for every non-zero  $f \in A$ . As  $A$  is finitely generated over  $D$ , so is  $A_f = A[T]/(1 - Tf)$ . Consider a factorization  $D_N \subseteq D_N[z_1, \dots, z_d] \subseteq A_{fN}$  as in Lemma 5.3. By the Lying Over Theorem,  $\text{Spec } A_{fN} \xrightarrow{\pi} \text{Spec } D_N$  is surjective. As  $\Gamma$  is dense,  $\Gamma \cap \text{Spec } D_N = \Gamma \cap \pi(\text{Spec } A_{fN})$  is non-empty. Consequently,  $\text{Spec } A_{fN} \cap \pi^{-1}(\Gamma)$ , and hence  $\text{Spec } A_f \cap \pi^{-1}(\Gamma)$ , is non-empty.  $\square$

### 5.3. Connections with $F$ -pure thresholds.

**Notation 5.6.** Let  $A$  be a finitely generated  $\mathbb{Z}$  sub-algebra of  $\mathbb{C}$ . We use  $S_A$  to denote the subring  $A[x_1, \dots, x_m] \subseteq S$ ; note that  $\mathbb{C} \otimes_A S_A = S$ . If  $\mu$  is a maximal ideal of  $A$ ,  $S_A(\mu)$  denotes the polynomial ring  $S_A \otimes_A A/\mu = S_A/\mu S_A = (A/\mu)[x_1, \dots, x_m]$ . By Corollary 5.4,  $\text{char } S_A(\mu) > 0$ . For  $g \in S_A$ ,  $g_\mu$  denotes the image of  $g$  in  $S_A(\mu)$ . Finally,  $\mathfrak{m}$  will denote the ideal generated by the variables  $x_1, \dots, x_m$  in the polynomial rings  $S, S_A$ , and  $S_A(\mu)$ .

It is an important fact that the  $F$ -pure (respectively, log canonical) threshold of a polynomial may also be defined in terms of its associated *test ideals* (respectively, *multiplier ideals*). Theorem 5.7 below was first observed in [MTW05, Theorem 3.4], and follows from deep theorems in [Smi00, HY03] relating test ideals and multiplier ideals. We refer the reader to the author's thesis for a detailed discussion of how to deduce Theorem 5.7 from the results of [Smi00, HY03].

**Theorem 5.7.** Let  $f \in S$  be a polynomial with  $f(\mathbf{0}) = 0$ . Then, for every finitely generated  $\mathbb{Z}$ -algebra  $A \subseteq \mathbb{C}$  with  $f \in S_A$ , the following hold:

- (1) There exists a dense open set  $U \subseteq \text{Spec } A$  such that  $\mathbf{fpt}_{\mathfrak{m}}(f_\mu) \leq \mathbf{lct}_{\mathbf{0}}(f)$  for every maximal ideal  $\mu \in U$ .

- (2) For every  $0 < \lambda < \mathbf{lct}_0(f)$ , there exists a dense open set  $U_\lambda \subseteq \operatorname{Spec} A$  such that  $\lambda \leq \mathbf{fpt}_m(f_\mu) \leq \mathbf{lct}_0(f)$  for every maximal ideal  $\mu \in U_\lambda$ .

We stress that the open set  $U_\lambda$  depends on  $\lambda$ , and often shrinks as  $\lambda$  increases.

**Remark 5.8.** Suppose that  $f$  has integer coefficients. If  $f_p$  denotes the image of  $f$  in  $\mathbb{F}_p[x_1, \dots, x_m]$ , the statements of Theorem 5.7 become

$$\mathbf{fpt}_m(f_p) \leq \mathbf{lct}_0(f) \text{ for } p \gg 0 \text{ and } \lim_{p \rightarrow \infty} \mathbf{fpt}_m(f_p) \leq \mathbf{lct}_0(f).$$

The behavior illustrated in Remark 5.8 is apparent in Example 0.1, which also shows that  $\mathbf{fpt}_m((x^2 + y^3)_p) = \mathbf{lct}_0(x^2 + y^3)$  for a dense, though not *open*, set of primes in  $\operatorname{Spec} \mathbb{Z}$ . This kind of behavior is of particular interest, and motivates the following definition.

**Definition 5.9.** Let  $f \in S$  with  $f(\mathbf{0}) = 0$ . We say *log canonicity equals dense  $F$ -purity* for  $f$  if for every finitely-generated  $\mathbb{Z}$ -algebra  $A \subseteq \mathbb{C}$  with  $f \in S_A$ , there exists a dense subset  $W \subseteq \operatorname{Spec} A$  such that  $\mathbf{fpt}_m(f_\mu) = \mathbf{lct}_0(f)$  for every maximal ideal  $\mu \in W$ . If  $W$  is open in  $\operatorname{Spec} A$ , we say that log canonicity equals *open  $F$ -purity* for  $f$ .

**Remark 5.10.** To show that log canonicity equals (open/dense)  $F$ -purity for  $f$ , it suffices to produce a *single* finitely-generated  $\mathbb{Z}$ -algebra  $A$  satisfying the conditions of Definition 5.9. We refer the reader to the author's thesis for a detailed verification of this.

**Remark 5.11.** In the study of *singularities of pairs*, the terms “log canonical” and “ $F$ -purity” have their own independent meanings. Indeed, one defines the notion of log singularities for *pairs*  $(S, \lambda \bullet f)$  via resolution of singularities (or via integrability conditions related to those in Definition 5.1) [Laz04]. Additionally, we have that

$$\mathbf{lct}_0(f) = \sup \{ \lambda > 0 : (S, \lambda \bullet f) \text{ is log canonical at } \mathbf{0} \},$$

which justifies the use of the term “log canonical threshold.” In the positive characteristic setting, one defines the notion of  $F$ -purity for pairs via the Frobenius morphism, and we again have that the  $F$ -pure threshold of a polynomial is the supremum over all parameters such that the corresponding pair is  $F$ -pure [HW02, TW04]. We say that the pair  $(S, \lambda \bullet f)$  is of *dense  $F$ -pure type* if for every (equivalently, for some) finitely generated  $\mathbb{Z}$ -algebra  $A \subseteq \mathbb{C}$  with  $f \in S_A$ , there exists a dense set  $W \subseteq \operatorname{Spec} A$  such that the pair  $(S_A(\mu), \lambda \bullet f_\mu)$  is  $F$ -pure for every maximal ideal  $\mu \in W$ . It is shown in [HW02, Tak04] that if  $(S, \lambda \bullet f)$  is log canonical, then it is also of dense  $F$ -pure type.

It is an important, yet easy to verify, property of log canonicity that  $(S, \mathbf{lct}_0(f) \bullet f)$  is log canonical at  $\mathbf{0}$ . Consequently,  $(S, \lambda \bullet f)$  is log canonical if and only if  $0 \leq \lambda \leq \mathbf{lct}_0(f)$ . In prime characteristic, that a pair is  $F$ -pure at the threshold is shown in [Har06, Her10b], and it follows that the reductions  $(S_A(\mu), \lambda \bullet f_\mu)$  are  $F$ -pure if and only if  $0 \leq \lambda \leq \mathbf{fpt}_m(f_\mu)$ . Examining the definitions, we reach the following conclusion: To show that log canonicity is equivalent to dense  $F$ -pure type for pairs  $(S, \lambda \bullet f)$ , it suffices to show that  $\mathbf{fpt}_m(f_\mu) = \mathbf{lct}_0(f)$  for all maximal  $\mu$  in some dense subset of  $\operatorname{Spec} A$ , which justifies our choice of terminology in Definition 5.9.

**Proposition 5.12.** [Fed83, Theorem 2.5] Let  $f \in S$  be a polynomial with  $\operatorname{Supp}(f) = \mathcal{M}$ , and let  $\alpha$  continue to denote the common value  $\mathbf{fpt}_m(\mathcal{M}) = \mathbf{lct}_0(\mathcal{M})$ . Let  $A \subseteq \mathbb{C}$  be a finitely-generated  $\mathbb{Z}$ -algebra such that  $f \in S_A$ . If  $\alpha > 1$  and  $f_\Lambda$  has an isolated singularity at  $\mathbf{0}$ , there exists a dense open subset  $U \subseteq \operatorname{Spec} A$  such that  $\mathbf{fpt}_m(f_\mu) = 1$  for every maximal ideal  $\mu \in U$ . In particular,  $\mathbf{lct}_0(f) = 1$ , and log canonicity equals open  $F$ -purity for  $f$ .

**Remark 5.13.** Proposition 5.12 generalizes [Fed83, Theorem 2.5]. By Remark 3.25, the assumption that  $f_\Lambda$  have an isolated singularity is unnecessary in dimension two.

*Proof of Proposition 5.12.* As discussed in Remark 3.25, it suffices to produce a single algebra  $A$  satisfying the conditions of the proposition. By hypothesis,

$$(5.3.1) \quad (x_1^N, \dots, x_m^N) \subseteq \left( \frac{\partial f_\Lambda}{\partial x_1}, \dots, \frac{\partial f_\Lambda}{\partial x_m} \right)$$

for some  $N \geq 1$ . Let  $A$  be the finitely-generated  $\mathbb{Z}$ -subalgebra of  $\mathbb{C}$  obtained by adjoining to  $\mathbb{Z}$  the coefficients of  $f$  and their inverses (so that  $\text{Supp } f_\mu = \mathcal{M}$  for every  $\mu \in \text{Spec } A$ ), and well as all of the coefficients needed to express every  $x_i^N$  as a linear combination of the partial derivatives of  $f_\Lambda$ . By construction,  $f \in S_A$ , and (5.3.1) also holds in  $S_A$ , and hence in  $S_A(\mu)$  for every maximal ideal  $\mu \in \text{Spec } A$ .

Let  $\mu \in \text{Spec } A$  be a maximal ideal. If  $\mathbf{fpt}_m(f_\mu) < 1$ , it follows from Lemma 3.23 that  $\text{char } A/\mu < N \cdot \frac{\alpha}{\alpha-1}$ . By Corollary 5.4,  $\{\mu : \text{char } A/\mu > N \cdot \frac{\alpha}{\alpha-1}\}$  is the set of all maximal ideals in some non-empty open set  $U_\circ$  of  $\text{Spec } A$ , and the first claim follows. Next, choose a dense open set  $U \subseteq \text{Spec } A$  satisfying the conditions of Theorem 5.7. For every  $\mu \in U_\circ \cap U$ ,  $1 = \mathbf{fpt}_m(f_\mu) \leq \mathbf{lct}_0(f) \leq \min\{1, \alpha\} = 1$ , and so we are done.  $\square$

**Lemma 5.14.** Let  $f \in S$  with  $\text{Supp}(f) = \mathcal{M}$ . Then, log canonicity equals dense  $F$ -purity for  $f$  if there exists a finitely-generated  $\mathbb{Z}$ -algebra  $A \subseteq \mathbb{C}$  with  $f \in S_A$ , an infinite set of primes  $\Gamma$ , and for every  $p \in \Gamma$  a set  $W_p$  satisfying the following conditions:

- (1)  $W_p$  is a dense in  $\pi^{-1}(p)$ , where  $\text{Spec } A \xrightarrow{\pi} \text{Spec } \mathbb{Z}$  denotes the map induced by  $\mathbb{Z} \subseteq A$ .
- (2)  $\mathbf{fpt}_m(f_\mu) = \min\{1, \mathbf{fpt}_m(\mathcal{M})\}$  for every maximal ideal  $\mu \in W_p$ .

*Proof.* Let  $W = \bigcup_{p \in \Gamma} W_p$ . As  $W_p$  is dense in  $\pi^{-1}(p)$ ,  $\overline{W_p} = \pi^{-1}(p)$ , and thus

$$(5.3.2) \quad \overline{W} \supseteq \bigcup_{p \in \Gamma} \overline{W_p} = \bigcup_{p \in \Gamma} \pi^{-1}(p) = \pi^{-1}(\Gamma).$$

By Corollary 5.5,  $\pi^{-1}(\Gamma)$  is dense in  $\text{Spec } A$ , and applying (5.3.2) shows that  $W$  is dense as well. Let  $U \subseteq \text{Spec } A$  be the dense open set given by Theorem 5.7. As  $W$  is dense and  $U$  is dense and open, it follows that  $U \cap W$  is dense in  $\text{Spec } A$ . Furthermore, for  $\mu \in U \cap W$ ,

$$(5.3.3) \quad \mathbf{lct}_0(f) \leq \min\{1, \mathbf{lct}_0(\mathcal{M})\} = \min\{1, \mathbf{fpt}_m(\mathcal{M})\} = \mathbf{fpt}_m(f_\mu) \leq \mathbf{lct}_0(f).$$

Indeed, the leftmost inequality in (5.3.3) holds by Proposition 5.2, the first equality by (5.1.2), the second equality by our assumption on  $W$ , and the rightmost inequality by the defining property of  $U$ . We conclude from (5.3.3) that  $\mathbf{fpt}_m(f_\mu) = \mathbf{lct}_0(f)$  for every maximal ideal  $\mu$  in the dense subset  $U \cap W$  of  $\text{Spec } A$ , and the claim follows.  $\square$

**Theorem 5.15.** If  $\mathbf{P}$  contains a unique maximal point  $\boldsymbol{\eta}$ , then log canonicity equals dense  $F$ -purity for every polynomial  $f \in S$  with  $\text{Supp}(f) = \mathcal{M}$ .

*Proof.* Let  $A$  be such that  $f \in S_A$  and consider the map  $\text{Spec } A \xrightarrow{\pi} \text{Spec } \mathbb{Z}$ . After enlarging  $A$ , we may assume that  $\text{Supp}(f_\mu) = \mathcal{M}$  for every  $\mu \in \text{mSpec } A$ . If  $|\boldsymbol{\eta}| \leq 1$ , let  $\Gamma$  consist of all primes  $p$  such  $\pi^{-1}(p)$  is non-empty, and such that the entries of  $\boldsymbol{\eta}$  add without carrying (in base  $p$ ). By Corollary 5.4 and Lemma 1.9,  $\#\Gamma = \infty$ , and it follows from Theorem 4.2

that  $\mathbf{fpt}_m(f_{\mu_p}) = |\eta| = \mathbf{fpt}_m(\mathcal{M})$  for all  $p \in \Gamma$  and  $\mu_p \in \pi^{-1}(p)$ . If  $|\eta| > 1$ , instead let  $\Gamma$  denote the set of all primes  $p$  such that  $\pi^{-1}(p)$  is non-empty and

$$(5.3.4) \quad \eta_1^{(e)} + \cdots + \eta_n^{(e)} \geq p \text{ for every } e \geq 1.$$

Again, Corollary 5.4 and Lemma 1.9 imply  $\#\Gamma = \infty$ , and (5.3.4) allows us to apply Theorem 4.2 with  $L = 0$ . Thus,  $\mathbf{fpt}_m(f_{\mu_p}) \geq 1$  for every  $p \in \Gamma$  and  $\mu_p \in \pi^{-1}(p)$ , and Remark 2.7 shows that equality must hold. If  $W_p := \pi^{-1}(p)$ , we see that  $A, \Lambda$ , and  $W_p$  satisfy the hypotheses of Lemma 5.14, and so we are done.  $\square$

**Theorem 5.16.** Log canonicity equals dense  $F$ -purity for every polynomial in  $S$  whose coefficients are algebraically independent over  $\mathbb{Q}$ .

*Proof.* We assume that  $f$  has support  $\mathcal{M}$  and algebraically independent coefficients  $u_1, \dots, u_n$ . If  $A := \mathbb{Z}[u_1, \dots, u_n]_{\prod u_i} \subseteq \mathbb{C}$ , then  $f \in S_A$  and  $\text{Supp}(f_\mu) = \mathcal{M}$  for all maximal ideals  $\mu \subseteq A$ . Set  $\gamma = \min\{1, \mathbf{fpt}_m(\mathcal{M})\}$ . By Proposition 3.7,  $\mathbf{fpt}_m(\mathcal{M}) = \max\{|\mathbf{s}| : \mathbf{s} \in \mathbf{P}\}$ , and we choose  $\lambda \in \mathbf{P}$  with  $|\lambda| = \gamma$ . By Remark 3.27, we may assume that  $\lambda$  has rational coordinates. If  $\Gamma$  denotes the set of primes  $p$  such that  $(p-1) \cdot \lambda \in \mathbb{N}^n$ , then  $\#\Gamma = \infty$  by Dirichlet's theorem on primes in arithmetic progressions. Fix a prime  $p \in \Gamma$ .

It follows from the monomial theorem that  $\mathbf{x}^{(p-1)\mathbf{E}\lambda}$  appears in  $f^{(p-1)\gamma}$  with coefficient

$$(5.3.5) \quad 0 \neq \Theta_{\lambda,p}(u_1, \dots, u_n) = \sum_{\substack{|\mathbf{k}|=(p-1)\gamma \\ \mathbf{E}\mathbf{k}=(p-1)\cdot\mathbf{E}\lambda}} \binom{(p-1) \cdot \alpha}{\mathbf{k}} \mathbf{u}^{\mathbf{k}} \in \mathbb{Z}[u_1, \dots, u_n] \subseteq A.$$

As  $\gamma \leq 1$ ,  $\binom{(p-1)\gamma}{\mathbf{k}} \neq 0 \pmod p$  for each  $\mathbf{k}$  in (5.3.5). By hypothesis,  $\mathbb{Z}[u_1, \dots, u_n]$  is a polynomial ring, and it follows that  $\Theta_{\lambda,p}(\mathbf{u})$  induces a non-zero element of the polynomial ring  $\mathbb{Z}/p\mathbb{Z}[u_1, \dots, u_n] \subseteq A/pA$ . Consider the map  $\text{Spec } A \xrightarrow{\pi} \text{Spec } \mathbb{Z}$  induced by the inclusion  $\mathbb{Z} \subseteq A$ . We have just shown that  $W_p := D(\Theta_{\lambda,p}(\mathbf{u})) \cap \pi^{-1}(p)$ , is a dense (open) subset of the fiber  $\pi^{-1}(p)$ . Let  $\mu_p$  be a maximal ideal in  $W_p$ . By definition, the image of  $\Theta_{\lambda,p}(\mathbf{u})$  is non-zero in  $A/\mu_p$ , and (5.3.5) shows that  $\mathbf{x}^{(p-1)\mathbf{E}\lambda}$  is contained in  $\text{Supp}\left((f_{\mu_p})^{(p-1)\gamma}\right)$  but not in  $\mathfrak{m}^{[p]}$  (as  $\lambda \in \mathbf{P}$ ). Thus,  $(f_{\mu_p})^{(p-1)\gamma} \notin \mathfrak{m}^{[p]}$ , which allows us to apply Lemma 2.5 (and Remark 2.7) to  $f_{\mu_p} \in S_A(\mu_p)$  to conclude that  $\mathbf{fpt}_m(f_{\mu_p}) = \gamma = \min\{1, \mathbf{fpt}_m(\mathcal{M})\}$ . We see that  $A, \Gamma$ , and  $W_p$  satisfy the conditions of Lemma 5.14, and so we are done.  $\square$

**Remark 5.17.** An important difference between Theorem 5.15 and Theorem 5.16 is that  $W_p = \pi^{-1}(p)$  in the former, while we only know that  $W_p \subseteq \pi^{-1}(p)$  in the latter. It would be interesting to investigate under what conditions  $F$ -pure thresholds remain constant over the fibers of certain distinguished primes in  $\text{Spec } \mathbb{Z}$ .

## REFERENCES

- [BL04] Manuel Blickle and Robert Lazarsfeld. An informal introduction to multiplier ideals. In *Trends in commutative algebra*, volume 51 of *Math. Sci. Res. Inst. Publ.*, pages 87–114. Cambridge Univ. Press, Cambridge, 2004. 2
- [BMS08] Manuel Blickle, Mircea Mustața, and Karen E. Smith. Discreteness and rationality of  $F$ -thresholds. *Michigan Math. J.*, 57:43–61, 2008. Special volume in honor of Melvin Hochster. 6
- [Dic02] L.E. Dickson. Theorems on the residues of multinomial coefficients with respect to a prime modulus. *Quarterly Journal of Pure and Applied Mathematics*, 33:378–384, 1902. 4



- [EM06] Lawrence Ein and Mircea Mustață. Invariants of singularities of pairs. In *International Congress of Mathematicians. Vol. II*, pages 583–602. Eur. Math. Soc., Zürich, 2006. 2, 15
- [Fed83] Richard Fedder.  $F$ -purity and rational singularity. *Trans. Amer. Math. Soc.*, 278(2):461–480, 1983. 2, 11, 17, 18
- [Har06] Nobuo Hara.  $F$ -pure thresholds and  $F$ -jumping exponents in dimension two. *Math. Res. Lett.*, 13(5-6):747–760, 2006. With an appendix by Paul Monsky. 17
- [Her10a] Daniel Jesús Hernández. An algorithm for computing  $F$ -pure thresholds of binomial hypersurfaces. *preprint*, 2010. 14
- [Her10b] Daniel Jesús Hernández.  $F$ -pure thresholds of hypersurfaces. *preprint*, 2010. 5, 17
- [Her10c] Daniel Jesús Hernández. On test ideals,  $F$ -pure thresholds, and higher  $F$ -jumping numbers of diagonal hypersurfaces. *preprint*, 2010. 11, 14
- [HH90] Melvin Hochster and Craig Huneke. Tight closure, invariant theory, and the Briançon-Skoda theorem. *J. Amer. Math. Soc.*, 3(1):31–116, 1990. 1
- [Hoc] Melvin Hochster. Supplemental Notes from Math 165 at the University of Michigan: Noether Normalization. <http://www.math.lsa.umich.edu/~hochster/615W10/supNoeth.pdf>. 16
- [How01] J. A. Howald. Multiplier ideals of monomial ideals. *Trans. Amer. Math. Soc.*, 353(7):2665–2671 (electronic), 2001. 15
- [HR76] Melvin Hochster and Joel L. Roberts. The purity of the Frobenius and local cohomology. *Advances in Math.*, 21(2):117–172, 1976. 1
- [HW02] Nobuo Hara and Kei-Ichi Watanabe.  $F$ -regular and  $F$ -pure rings vs. log terminal and log canonical singularities. *J. Algebraic Geom.*, 11(2):363–392, 2002. 1, 2, 17
- [HY03] Nobuo Hara and Ken-Ichi Yoshida. A generalization of tight closure and multiplier ideals. *Trans. Amer. Math. Soc.*, 355(8):3143–3174 (electronic), 2003. 1, 2, 3, 16
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals. 2, 17
- [Luc78] Edouard Lucas. Theorie des Fonctions Numeriques Simplement Periodiques. *Amer. J. Math.*, 1878. 4
- [MTW05] Mircea Mustață, Shunsuke Takagi, and Kei-ichi Watanabe.  $F$ -thresholds and Bernstein-Sato polynomials. In *European Congress of Mathematics*, pages 341–364. Eur. Math. Soc., Zürich, 2005. 1, 2, 5, 16
- [Sch08] Karl Schwede. Generalized test ideals, sharp  $F$ -purity, and sharp test elements. *Math. Res. Lett.*, 15(6):1251–1261, 2008. 5
- [Smi97] Karen E. Smith. Vanishing, singularities and effective bounds via prime characteristic local algebra. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 289–325. Amer. Math. Soc., Providence, RI, 1997. 2
- [Smi00] Karen E. Smith. The multiplier ideal is a universal test ideal. *Comm. Algebra*, 28(12):5915–5929, 2000. Special issue in honor of Robin Hartshorne. 2, 16
- [Tak04] Shunsuke Takagi.  $F$ -singularities of pairs and inversion of adjunction of arbitrary codimension. *Invent. Math.*, 157(1):123–146, 2004. 2, 17
- [TW04] Shunsuke Takagi and Kei-ichi Watanabe. On  $F$ -pure thresholds. *J. Algebra*, 282(1):278–297, 2004. 1, 5, 17
- [Web94] Roger Webster. *Convexity*. Oxford Science Publications. The Clarendon Press Oxford University Press, New York, 1994. 7, 8